

NUMBERS



MODULAR SYSTEM

Zambak
PUBLISHING

M A T H E M A T I C S S E R I E S

MODULAR SYSTEM

NUMBERS

Ertuğrul Tarhan
Cem Giray



<http://book.zambak.com>

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PREFACE

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To the Teacher

This book covers the syllabus of a high-school course in elementary number theory. It builds upon students' knowledge of basic number properties and leads them to a greater understanding of real numbers, including exponential and radical expressions. Before beginning this book, students should have successfully completed a pre-algebra course in the basic principles of number sets, arithmetical operations, factorization, and exponential and radical notation.

The book is divided into two chapters, structured as follows:

- ◆ *Chapter 1 covers number sets. It begins with the study of natural numbers and the properties of divisibility, and continues with coverage of integer and rational numbers. Finally, students study the set of real numbers. An optional section at the end of this chapter presents modular arithmetic and its applications.*
- ◆ *Chapter 2 builds on the material of the previous chapter and covers exponential and radical expressions. Taking the concept of square root as a starting point, the chapter leads students to an understanding of operations with any rational power.*

Number theory is a vast topic, and it is obviously impossible to cover every type of question in the text. However, we believe that we have provided a sufficient range of explanations, algorithms and examples in the text to enable students to approach most problems in elementary number theory.

This book has been designed to be an effective teaching aid, and includes all of the features of the Zambak Modular System high school math teaching series:

- ◆ *The book uses a linear teaching approach, with material in the latter sections building on concepts and math covered previously in the text.*
- ◆ *Self-test 'Check Yourself' sections at key points in the text allow students to check their understanding, and build students' confidence for further study.*
- ◆ *Exercises at the end of each section give students a chance to use the skills and techniques they have learned to solve both pure and applied problems.*
- ◆ *A chapter summary at the end of each chapter provides a concise review of the main content of the chapter. Included in the summary are a set of concept check questions which ask students to explain key concepts and ideas in their own words.*
- ◆ *Review tests cover the material in a whole chapter, and help to prepare students for exams.*

Acknowledgements

Many friends and colleagues were of great help in writing this textbook. We would like to thank everybody who helped us at Zambak Publications, especially Mustafa Kırıkçı, Muhammer Taşkiran and Gafur Taşkın. Special thanks also go to Serdar Çam and Serdal Yıldırım for their patient typesetting and design.

The Authors

This book is designed so that you can use it effectively. Each chapter has its own special color that you can see at the bottom of the page.

Chapter 1

Real Numbers

Chapter 2

Exponents and Radicals

principal square

If $b^2 = a$, then
 say

Let M be a com-
folle

Notation

The symbol $\sqrt{\quad}$

Check Yourself 2

- Evaluate the expressions.
 - 5^3
 - 3^5
- $m = 3^n$ is given

A small notebook in the left or right margin of a page reminds you of the math you need so that you can understand the material. It might help you to see your mistakes, too! Notebooks are the same color as the section you are studying.

< means 'is less than'
> means 'is greater than'
≤ means 'is less than or equal to'

PROPERTIES OF POWERS
For all $a, b, m, n \in \mathbb{N}$
PROPERTIES OF SQUARE ROOTS
For any $a \geq 0, b \geq 0$ and $n \in \mathbb{Z}$

Special boxes highlight important new information. These boxes may contain formulas, properties, and solution procedures, etc. They are the same color as the section you are studying.

Exercises at the end of each section cover the material in the whole section. You should be able to solve all the problems which do not have a star. One star (*) next to a question means the question is a bit more difficult. Two stars (***) next to a question means the question is for students who are looking for a challenge! The answers to the exercises are at the back of the book.

EXERCISES 1.1
A. Natural Numbers
1. Determine whether each of the natural number or not.
a. $[20 - (3 - 5)]$
b.

CHAPTER SUMMARY
Natural Numbers
Concept Check
What is the difference between a set of digits?
CHAPTER REVIEW TEST
1. Evaluate $3^2 - 2^2 + 3^2 - 3^2 + 2^2$
A) 49

The **Chapter Summary** at the end of each chapter summarizes all the important material that has been covered in the chapter. The **Concept Check** section contains oral questions. In order to answer them you don't need paper or pen. If you answer Concept Check questions correctly, it means you know that topic! The answers to these questions are in the material you studied. Go back over the material if you are not sure about an answer to a Concept Check question. Finally, the **Chapter Review Tests** include multiple choice questions in increasing order of difficulty to help you prepare for exams. The answer key for these tests is at the back of the book.

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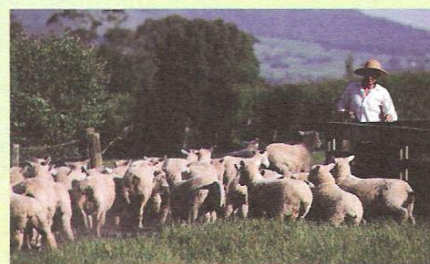
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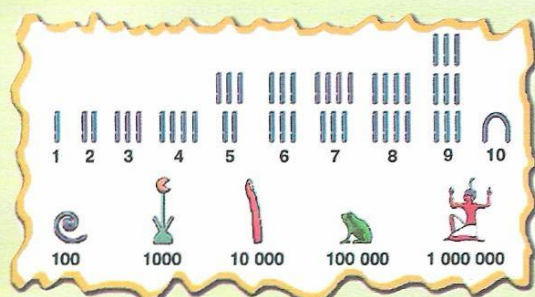
INTRODUCTION

Throughout the history of mankind, people have needed to count. In very ancient times, a shepherd carried a bag of pebbles or stones to help count his sheep. In the bag there was one pebble for each sheep. By matching each pebble to a sheep, the shepherd could see if any of his sheep were missing. However, a shepherd could not say how many sheep he had because there were no words for numbers. He could only show the bag of pebbles.



Later on, people began to use words instead of things to answer the question 'how many?'. For example, some people used the word for 'hand' to mean five things. They used the name of a flower with four leaves to mean four things, and so on. Every primitive civilization had a different set of words like this, but the idea was the same. People recognized that there was something similar between, for example, five sheep, five pebbles and a hand. This was the beginning of numbers.

Finally, people began to put the names of the numbers in order so that they could count things. Instead of using a bag of pebbles, a shepherd could say the word for 'one', followed by the word for 'two', then the word for 'three', and so on. This was the beginning of counting systems, also called number systems. Over time, different people in different parts of the world developed different counting systems.



Five thousand years ago, the Egyptians used a special set of symbols called hieroglyphics to represent the units, tens, hundreds and thousands, etc. in their number system. Today we call the Egyptian system a decimal system, because it is based on the number ten. The Egyptian system used seven different symbols to represent the numbers 1, 10, 100, 1000, 10 000, 100 000 and 1 000 000.

In around 450 BC, the Greeks were also counting with a decimal system, but they used the letters of the alphabet for the different numbers. They used the first eight letters of the alphabet to represent the numbers 1 to 8 and a special symbol for the number nine. There was no symbol for zero.

The Romans also used letters in a decimal system. In this system, the number five and multiples of five were important, so the symbols were I (for 1), V (for 5), X (for 10), L (for 50), C (for 100), D (for 500) and M (for 1000). For example, the number 1863 was written as MDCCCLXIII in the Roman system.

Not all number systems were based on the number ten, however. In about 300 BC, the Babylonians used a system based on the number 60 (called a hexadecimal system). This system is interesting because it used the concept of position, similar to the modern system we use today.

In most of these systems, the numbers were written horizontally. However, other civilizations such as the Chinese and the Mayans wrote their numbers vertically. The Chinese system was a decimal system, but the Mayan system was based on the numbers 20 and 360.

One very important milestone in the history of numbers was the introduction of the number zero. The muslim mathematician Muhammad Bin Musa al-Khwarizmi (A.D. 780 - 850) was the first person to use zero as a placeholder in positional base notation.

The modern number system which we use today is called the Hindu-Arabic system, because it was invented by the Hindus and then introduced into Europe by the Arabs between 700 and 1100 AD. This system did not always use the number symbols we use today. You can see its development in the chart below.

1	2	3	4	5	6	7	8	9	0	→	Modern Arabic (Western)
١	٢	٣	٤	٥	٦	٧	٨	٩	٠	→	Early Arabic (Western)
ا	ب	ج	د	هـ	و	ز	ح	ط	ق	→	Arabic letters (used as numerals)
١	٢	٣	٤	٥	٦	٧	٨	٩	.	→	Modern Arabic (Eastern)
١	٢	٣	٤	٥	٦	٧	٨	٩	.	→	Early Arabic (Eastern)
१	२	३	४	५	६	७	८	९	०	→	Later Devanagari (Hindu)
१	२	३	४	५	६	७	८	९	०	→	Early Devanagari (Hindu)

As mathematicians worked with numbers, they began to divide them into different groups. It was easy to group the numbers 1, 2, 3 etc. into one group, which was called the set of counting numbers. Some mathematicians added zero to this group to form the set of natural numbers, although even today, not all mathematicians agree that zero is a natural number. The counting numbers, their negative equivalents (-1, -2, -3, etc.) and zero together form another set: the integers.

Of course, people did not only want to count whole objects with numbers. Sometimes they wanted to know or express a ratio, or count only a fraction of a whole. For this reason, rational numbers (i.e. numbers which can be expressed as a ratio of integers) have been used since the earliest number systems. But the set of integer ratios does not include important numbers such as $\sqrt{2}=1.414213...$, $\pi=3.14159...$ and $e=2.71828...$. These numbers, which cannot be expressed as an integer ratio, form the set of irrational numbers. The rational numbers and irrational numbers together form the set of real numbers, which is the set of numbers you have worked with in your algebra studies so far.

This book describes the set of real numbers and the main properties of the numbers in each of its subsets: natural numbers, integers, rational numbers and irrational numbers. As you work through the book you will probably recognize some of the math you covered in your previous studies of numbers and their properties, but many of the concepts and proofs here will be new to you. Number theory is one of the main building blocks of algebra and many other parts of mathematics, so take time to study the material in this book carefully. We wish you every success in your studies.



Chapter 1

REAL NUMBERS

1

NATURAL NUMBERS

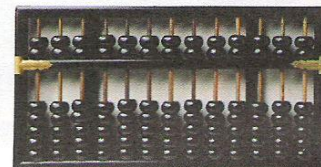
A. NATURAL NUMBERS

1. Counting Numbers, Natural Numbers and Whole Numbers



Mathematicians sometimes disagree about whether zero is a natural number or a whole number. In this book, zero is not a natural number but it is a whole number.

We live in a world of numbers. We use numbers in many different ways, but the simplest use is for counting. When we count the number of items in a group, we begin with the number 1 and continue with 2, 3, 4 and so on. For this reason, we call the numbers 1, 2, 3, 4, etc. **counting numbers**. In this book, the set of counting numbers is also called the set of **natural numbers** (denoted by $\mathbb{N}$). The set of whole numbers is the set of natural numbers plus the number zero.



An abacus is a counting and calculation tool.

Definition

natural number, counting number, whole number

The set of **natural numbers** (also called the set of **counting numbers**) is denoted by $\mathbb{N}$:

$$\mathbb{N} = \{1, 2, 3, \dots, n, n + 1, \dots\}.$$

Natural numbers together with zero are called **whole numbers**. The set of whole numbers is denoted by $\mathbb{W}$.

EXAMPLE

1 The sum of four consecutive natural numbers is 38. Find the numbers.

Solution

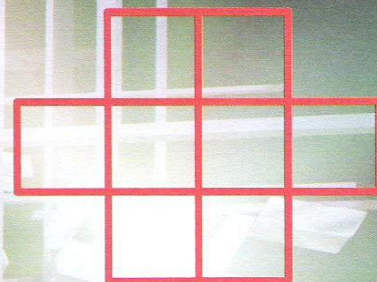
Let the first number be n . Then the next three numbers will be $n + 1$, $n + 2$ and $n + 3$.

So $n + n + 1 + n + 2 + n + 3 = 38$, which means $4n + 6 = 38$, i.e. $n = 8$.

Therefore, the numbers are 8, 9, 10 and 11.

PUZZLE

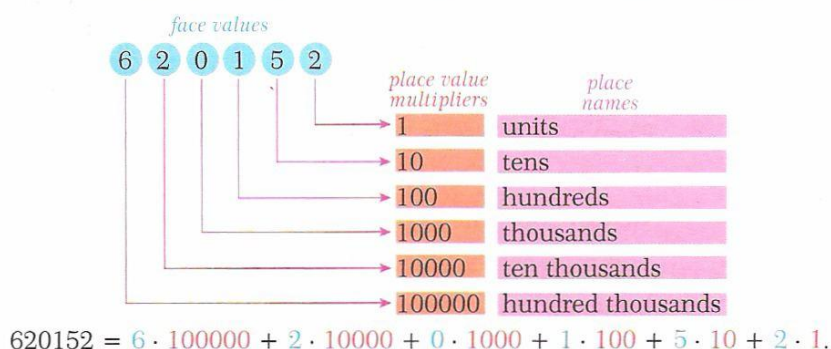
Place the numbers 1, 2, 3, 4, 5, 6, 7 and 8 in the cells so that consecutive numbers are not adjacent to each other.



2. Digits of a Number

The modern numeration system that we use is called the **decimal system**. It takes its name from the Latin word *decem*, which means 'ten'. In the decimal system, the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are used to represent a natural number. Each of the symbols in a number is called a **digit**. For example, 3640 is a number with four digits, i.e. it is a four-digit number.

Each digit in a number has a **face value**, which is the value of the symbol used. Each digit also has a **place value**, which is the product of the face value by the number 1, 10, 100 and so on from the rightmost digit to the left, respectively. For example, in the number 3640 the face value of the digit '6' is six and its place value is $6 \cdot 100 = 600$. Look at another example:



EXAMPLE 2 A two-digit natural number is equal to twice the sum of its digits. Find the number.

Solution Let ab be the two-digit number, where a and b are digits.

$$ab = 2 \cdot (a + b)$$

$$(10 \cdot a) + b = (2 \cdot a) + (2 \cdot b)$$

$$8 \cdot a = b$$

Since a and b can only have a value between 0 and 9 inclusive, we get $a = 1$ and $b = 8$.

So $ab = 18$.

In this example, note that ab stands for a two-digit number, not the product $a \cdot b$.

Notation

Unless otherwise stated, ab denotes the product $a \cdot b$.

If it is given that ab is a two-digit number, then a and b are digits.

Some books use the notation $\overline{ab}$ to denote a two-digit number with digits a and b .

Note

If ab is a two-digit number then $ab = 10a + b$.

If abc is a three-digit number then $abc = 100a + 10b + c$.

3. Even and Odd Numbers

Definition

even number, odd number

If a natural number is a multiple of 2 then it is called an **even** number, otherwise it is an **odd** number.

We can list the set of even numbers as $\{2, 4, 6, \dots, 2n, 2n + 2, \dots\}$.

We can list the set of odd numbers as $\{1, 3, 5, \dots, 2n - 1, 2n + 1, \dots\}$.

EXAMPLE

3

Prove the statements.

- The sum of two even numbers is even.
- The sum of two odd numbers is even.

Solution

- Let n_1 and n_2 be two even numbers.

Then $n_1 = 2k_1$ ($k_1 \in \mathbb{N}$) and $n_2 = 2k_2$ ($k_2 \in \mathbb{N}$).

Hence $n_1 + n_2 = 2k_1 + 2k_2 = 2(k_1 + k_2)$.

We can now write $k_3 = k_1 + k_2 \in \mathbb{N}$, so $n_1 + n_2 = 2k_3$.

This is an even number, which proves the statement.

- Let m_1 and m_2 be two odd numbers. Then $m_1 = 2k_1 - 1$ ($k_1 \in \mathbb{N}$) and $m_2 = 2k_2 - 1$ ($k_2 \in \mathbb{N}$).

Hence $m_1 + m_2 = (2k_1 - 1) + (2k_2 - 1) = 2(k_1 + k_2 - 1)$, and $k_1 + k_2 - 1 = k_3 \in \mathbb{N}$.

Therefore the sum $m_1 + m_2 = 2k_3$ is an even number.

Seven is an odd number.
How can we make it even?
Take away the 5!
Then it will be even.



Note

Let E denote an even number and O denote an odd number. Then the following statements are true.

1. $E + E = E$

2. $O + O = E$

3. $E + O = O$

4. $E \cdot E = E$

5. $O \cdot O = O$

6. $O \cdot E = E$

EXAMPLE

4

Given that $5(2a - b)$ is even and $7a + 11$ is even, determine whether a and b are even or odd.

Solution

Since 5 is an odd number and $5(2a - b)$ is even, $2a - b$ must be even ($O \cdot E = E$). But $2a$ is even for any value of a , so b must also be even.

Since 11 is an odd number and $7a + 11$ is even, $7a$ must be odd ($O + O = E$). If $7a$ is odd then a is also odd ($O \cdot O = O$).

Check Yourself 1

- ab , ba and $c4$ are three two-digit numbers such that $ab + ba = c4$. Find $a + b + c$.
- a and b are even and c is odd. Determine whether $2(5a - b) + 3c - 7$ is even or odd.

Answers

- 8
- even

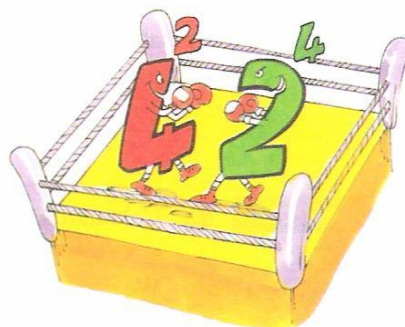


4. Powers of Natural Numbers

Consider the number $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$. We can write this number more simply as 5^8 . 5^8 is an **exponential expression** which means 5 multiplied by itself eight times.

Similarly we can write $a \cdot a = a^2$, $a \cdot a \cdot a = a^3$, and $a \cdot a \cdot a \cdot a = a^4$, etc.

In general, $a \cdot a \cdot a \cdot \dots \cdot a = a^n$, which means a multiplied by itself n times. Do not confuse a^n with $na = a + a + \dots + a$, which is the sum of n terms.



Definition

power of a number, base, exponent

For any natural number a and real number n ,

$$a^n = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

In this notation, a^n is called a **power** of a . a is called the **base** and n is called the **exponent**. We read a^n as ' a to the n th power'.

For example, $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$,

$$11^3 = 11 \cdot 11 \cdot 11 = 1331,$$

$$719^1 = 719, \text{ and } 1^{2007} = 1.$$

Which expression is greater: $(2^{90} + 2^{90})$ or 2^{100} ? Can you explain why?

Note

Let a , m and n be natural numbers. Then the following statements are true.

1. $a^1 = a$
2. $1^m = 1$
3. $a^m = a^n$ if and only if $m = n$.

PROPERTIES OF POWERS OF NATURAL NUMBERS

For all $a, b, m, n \in \mathbb{N}$:

$$1. a^m \cdot a^n = a^{m+n} \quad 2. (a \cdot b)^m = a^m \cdot b^m \quad 3. (a^m)^n = a^{m \cdot n}.$$

Proof

$$1. a^m \cdot a^n = \underbrace{(a \cdot a \cdot a \cdot \dots \cdot a)}_{m \text{ times}} \cdot \underbrace{(a \cdot a \cdot a \cdot \dots \cdot a)}_{n \text{ times}} = \underbrace{(a \cdot a \cdot a \cdot \dots \cdot a)}_{m+n \text{ times}} = a^{m+n}$$

$$2. (a \cdot b)^m = \underbrace{(a \cdot b) \cdot (a \cdot b) \cdot \dots \cdot (a \cdot b)}_{m \text{ times}} = \underbrace{(a \cdot a \cdot a \cdot \dots \cdot a)}_{m \text{ times}} \cdot \underbrace{(b \cdot b \cdot b \cdot \dots \cdot b)}_{m \text{ times}} = a^m \cdot b^m$$

$$3. (a^m)^n = \underbrace{a^m \cdot a^m \cdot a^m \cdot \dots \cdot a^m}_{n \text{ times}} = a^{\overbrace{m+m+\dots+m}^{n \text{ times}}} = a^{m \cdot n}$$

EXAMPLE 5 $a^n = 3^9$ and $a^m = 3^6$ are given. Find the value of a^{2n+m} .

Solution

$$\begin{aligned} a^{2n+m} &= a^{2n} \cdot a^m && \text{(by property 1)} \\ &= (a^n)^2 \cdot a^m && \text{(by property 3)} \\ &= (3^9)^2 \cdot 3^6 = 3^{18} \cdot 3^6 = 3^{18+6} = 3^{24}. \end{aligned}$$

EXAMPLE 6 Calculate $81^5 \cdot 243^7$, given that $81 = 3^4$ and $243 = 3^5$.

Solution We can substitute $81 = 3^4$ and $243 = 3^5$ into the expression:

$$81^5 \cdot 243^7 = (3^4)^5 \cdot (3^5)^7 = (3^{4 \cdot 5}) \cdot (3^{5 \cdot 7}) = 3^{20} \cdot 3^{35} = 3^{20+35} = 3^{55}.$$

EXAMPLE 7 a and m are two natural numbers. Write the product $(3 \cdot m \cdot a^3)^4 \cdot (27 \cdot a^4)^3$ in its simplest form.

Solution

$$\begin{aligned} (3 \cdot m \cdot a^3)^4 \cdot (27a^4)^3 &= [3^4 \cdot m^4 \cdot (a^3)^4] \cdot [(27)^3 \cdot (a^4)^3] && \text{(by property 2)} \\ &= [3^4 \cdot m^4 \cdot a^{3 \cdot 4}] [(3^3)^3 \cdot a^{4 \cdot 3}] && \text{(by property 3)} \\ &= [3^4 \cdot m^4 \cdot a^{12}] [3^9 \cdot a^{12}] \\ &= (3^4 \cdot 3^9)(a^{12} \cdot a^{12})m^4 \\ &= 3^{13} \cdot a^{24} \cdot m^4 && \text{(by property 1)} \end{aligned}$$

EXAMPLE 8

- $3^{3a+1} = m$ is given. Find the value of 27^{a+1} in terms of m .
- Find $n \in \mathbb{N}$ such that $(9^n)^{2n} = 3^{100}$.

Solution

- $27^{a+1} = (3^3)^{a+1} = 3^{3a+3} = 3^{3a+1} \cdot 3^2 = m \cdot 9 = 9m$
- $$\begin{aligned} (9^n)^{2n} &= 3^{100} \\ [(3^2)^n]^{2n} &= 3^{100} \\ 3^{4n^2} &= 3^{100} \\ 4n^2 &= 100 \end{aligned}$$

Since n is a natural number, $n = 5$.

EXAMPLE 9 How many digits does the number $616 \cdot 2^{21} \cdot 5^{22}$ have?

Solution To find the number of digits, we need to express the number as a power of 10.

$$\begin{aligned} 616 \cdot 2^{21} \cdot 5^{22} &= 308 \cdot 2 \cdot 2^{21} \cdot 5^{22} = 308 \cdot 2^{22} \cdot 5^{22} = 308 \cdot (2 \cdot 5)^{22} = 308 \cdot 10^{22} \\ 308 \cdot 10^{22} &\text{ means 308 followed by 22 zeros. So the number has 25 digits.} \end{aligned}$$


EXAMPLE 10 Calculate $\frac{2^{2005}}{16^{501}}$.

Solution $\frac{2^{2005}}{16^{501}} = \frac{2^{2005}}{(2^4)^{501}} = \frac{2^{2005}}{2^{2004}} = \frac{2^{2004} \cdot 2}{2^{2004}} = 2$

EXAMPLE 11 Which number is greater: 2^{30} or 3^{20} ?

Solution $2^{30} = (2^3)^{10} = 8^{10}$ and $3^{20} = (3^2)^{10} = 9^{10}$

Since we have the **same exponent** and 9 is greater than 8, 3^{20} is greater than 2^{30} .

Check Yourself 2

1. Evaluate the expressions.

a. 5^3 b. 3^5 c. 2^8

2. $m = 3^n$ is given. Express 81^{2n+5} in terms of m .

3. How many digits does the number $1680 \cdot 2^{20} \cdot 5^{23}$ have?

4. $A = \underbrace{999\dots9}_{m \text{ times}}$ and $A = 10^{50} - 1$ are given. Find m .

Answers

1. a. 125 b. 243 c. 256 2. $3^{20} \cdot m^8$ 3. 26 4. 50

MAGIC NUMBERS

$$6 \times 7 = 42$$

$$66 \times 67 = 4422$$

$$666 \times 667 = 444222$$

$$6666 \times 6667 = 44442222$$

$$3^3 + 7^3 + 1^3 = 371$$

$$1^3 + 5^3 + 3^3 = 153$$

$$1 \times 9 + 2 = 11$$

$$12 \times 9 + 3 = 111$$

$$123 \times 9 + 4 = 1111$$

$$1234 \times 9 + 5 = 11111$$

$$(5 + 8 + 3 + 2)^3 = 5832$$

$$(1 + 7 + 5 + 7 + 6)^3 = 17576$$

$$(1 + 9 + 6 + 8 + 3)^3 = 19683$$

5. The Division Algorithm

Let a and b be two natural numbers. We say that a is **divisible by** b if and only if the remainder after dividing a by b in the set of natural numbers is zero. For example, since $36 \div 9$ has remainder zero, 36 is divisible by 9. However, $6 \div 4$ has remainder 2 in the set of natural numbers, so 6 is not divisible by 4. We can give the formal definition of divisibility as follows:

Definition

divisibility

Let a and b be two natural numbers. If there is a natural number c which satisfies $b \cdot c = a$ then we say that a is **divisible by** b , or b **divides** a .

If a is divisible by b then b is called a **factor** (or divisor) of a , and a is called a **multiple** of b . For example, 5 is a factor of 30, and 30 is a multiple of 5. The number 30 can be written as the product of two of its factors, for example 5 and 6. The expression $5 \cdot 6$ is called a **factorization** of 30. ($3 \cdot 10$) is another factorization of 30.

If a number is not divisible by a certain number then there will be a remainder. For example, the number 43 is not divisible by 5. When we try to divide 43 by 5, the result is a number between 8 and 9 since $(5 \cdot 8) < 43 < (5 \cdot 9)$. We can write $43 = (5 \cdot 8) + 3$.

In this equation, 43 is called the **dividend**, 5 is the **divisor**, 8 is the **quotient** and 3 is the **remainder**. The remainder after division is always less than the divisor.



< means 'is less than'.
> means 'is greater than'.
≤ means 'is less than or equal to'.
≥ means 'is greater than or equal to'.

THE DIVISION ALGORITHM

$$\begin{array}{r|l} \text{dividend } a & \text{divisor } b \\ \hline b \cdot c & \text{quotient } c \\ \hline r & \text{remainder } r \end{array} \quad a = (b \cdot c) + r, \text{ where } 0 \leq r < b$$

When the remainder is zero, a is divisible by b .

EXAMPLE

12

Find the remainders when the natural numbers 32, 33, 34, 35, 36 and 37 are each divided by 3 and 4.

Solution

When a natural number is divided by 3, the remainder r must satisfy the inequality $0 \leq r < 3$.

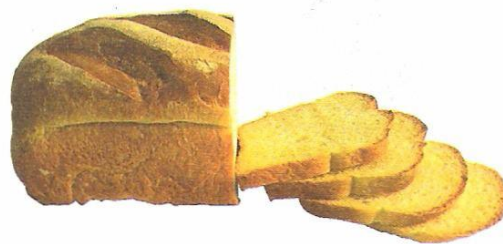
When a natural number is divided by 4, the remainder r satisfies $0 \leq r < 4$.

We can write the remainders as follows:

$32 = 3 \cdot 10 + 2, \quad r = 2$	$32 = 4 \cdot 8 + 0, \quad r = 0$
$33 = 3 \cdot 11 + 0, \quad r = 0$	$33 = 4 \cdot 8 + 1, \quad r = 1$
$34 = 3 \cdot 11 + 1, \quad r = 1$	$34 = 4 \cdot 8 + 2, \quad r = 2$
$35 = 3 \cdot 11 + 2, \quad r = 2$	$35 = 4 \cdot 8 + 3, \quad r = 3$
$36 = 3 \cdot 12 + 0, \quad r = 0$	$36 = 4 \cdot 9 + 0, \quad r = 0$
$37 = 3 \cdot 12 + 1, \quad r = 1$	$37 = 4 \cdot 9 + 1, \quad r = 1$



Note that the set of possible remainders when a natural number is divided by 3 is $\{0, 1, 2\}$. A similar set for division by 4 is $\{0, 1, 2, 3\}$. Which remainders can we have when a natural number is divided by 5?



EXAMPLE 13 Find all the possible values of b in the division operation.

$$\begin{array}{r} 15 \overline{) \frac{b}{c}} \\ \underline{} \\ 6 \end{array}$$

Solution We know that $15 = (b \cdot c) + 6$ and $b > 6$.

Therefore we are looking for a natural number b such that $bc = 9$ and $b > 6$.

Since c is a natural number, the only possible value of b is 9.

EXAMPLE 14 When a natural number p is divided by 3 its remainder is 1. Find the remainder when the square of p is divided by 3.

Solution We can write $p = 3n + 1$, where n is a natural number.

$$\text{So } p^2 = (3n + 1)^2 = 9n^2 + 6n + 1 = 3(3n^2 + 2n) + 1.$$

Since $3n^2 + 2n$ is a natural number and $1 < 3$, the remainder is 1.

EXAMPLE 15 abc and cba are two three-digit numbers. Show that $abc - cba$ is divisible by 11.

Solution $abc - cba = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c = 11(9a - 9c)$

Since 11 is a factor of $abc - cba$, this number is divisible by 11.

Check Yourself 3

1. $n \in \mathbb{N}$ and $\begin{array}{r} A \overline{) \frac{7}{n-1}} \\ \underline{} \\ n+1 \end{array}$ are given. Find the maximum possible value of A .

2. Find the possible values of $n \in \mathbb{N}$ if $\frac{3n+21}{n+3}$ is a natural number.

3. abc and bca are two three-digit numbers. Show that $abc - bca$ is divisible by 9.

Answers

1. 34 2. $n \in \{1, 3, 9\}$ 3. Hint: Write the difference so that 9 is a factor.

MIND READING

It is late at night on a television magic show.

HOST: Ladies and gentlemen! It's time to get ready for the final trick in our magic show tonight. Please give a big warm welcome to The Great Mind Reader, who will tell you what's 'on your mind' this evening! (*Applause. The MIND READER walks onto the stage.*)

MIND READER: (*concentrating and looking at the audience*) Oh! Oh! I hear voices ... Too many voices! I need a volunteer ... Yes! You, sir! ... Come here, please. (*A VOLUNTEER from the audience comes forward to join the Mind Reader.*) But no, no! One mind is too easy for me tonight. One more volunteer, please ... Yes, you, sir! I shall read two minds at the same time! (*Another VOLUNTEER takes the stage.*)

HOST: (*to the Volunteers*) Gentlemen! Thank you for your participation! Let the magic begin! Please think of a number between 0 and 9 and write it on a piece of paper. Do not tell us your number!

The Volunteers look at each other suspiciously. Then they each take a piece of paper and write something on it: Volunteer 1 secretly writes the number 7, and Volunteer 2 secretly writes the number 3.

VOLUNTEERS (*together*): OK.

HOST: Ladies and gentlemen! The Great Mindreader will now read the minds of our volunteers tonight! Silence, please!

MIND READER: (*to Volunteer 1*) Sir, please multiply your number by 2!

VOLUNTEER 1: (*thinking silently: $7 \cdot 2 = 14$*)

MIND READER: (*to Volunteer 1*) Add 5 to the result.

VOLUNTEER 1: (*thinking silently: $14 + 5 = 19$*)

MIND READER: (*to Volunteer 1*) Multiply your result by 5!

VOLUNTEER 1: (*thinking silently: $19 \cdot 5 = 95$*)

MIND READER: (*to Volunteer 2*) Sir, please add 10 to your number!

VOLUNTEER 2: (*thinking silently: $3 + 10 = 13$*)

MIND READER: (*trembling as he talks to the volunteers*) Oh! Oh! I am so close ... Please ... add your two results and tell me the answer!

Volunteer 1 and Volunteer 2 discuss something briefly.

VOLUNTEER 1: (*thinking silently: $95 + 13 = 108$*) 108!

MIND READER: (*thinking silently: $108 - 35 = 73$, so Volunteer 1's number is 7 (the first digit) and Volunteer 2's number is 3 (the second digit)*) Yes! ... Yes! ... The numbers are ... 7 ... and 3!

Is The Great Mind Reader really reading minds? What is so special about the number 35? Try the Mind Reader's trick on your friends! Can you see how it works?

B. PRIME NUMBERS

Most natural numbers can be factored in more than one way. Consider the natural number 24.

We can write:

$$24 = 1 \cdot 24$$

$$24 = 2 \cdot 12$$

$$24 = 3 \cdot 8$$

$$24 = 4 \cdot 6$$

$$24 = 2 \cdot 2 \cdot 2 \cdot 3.$$

We can see that the set of all factors of 24 is $\{1, 2, 3, 4, 6, 8, 12, 24\}$.

In a similar way,

the set of all factors of 8 is $\{1, 2, 4, 8\}$,

the set of all factors of 6 is $\{1, 2, 3, 6\}$,

the set of all factors of 11 is $\{1, 11\}$,

the set of all factors of 18 is $\{1, 2, 3, 6, 9, 18\}$ and

the set of all factors of 23 is $\{1, 23\}$.

Notice that 23 has only two factors: 23 and 1. Similarly, 11 has only two factors: itself and 1.

We call numbers like this **prime numbers**.

Definition

prime number, composite number

A **prime number** p is a natural number that has only two factors, 1 and p . A natural number which has more than two factors is called a **composite number**.

We can also define a prime number as a natural number whose set of factors contains exactly two elements. For example,

1 is not prime since the only factor of 1 is 1 (only one element).

2 is prime since the only factors of 2 are 1 and 2 (two elements).

3 is prime since the only factors of 3 are 1 and 3.

4 is not prime, since the factors of 4 are 1, 2 and 4 (three elements). So 4 is a composite number.

5 is prime since the only factors of 5 are 1 and 5.

6 is not prime, since the factors of 6 are 1, 2, 3 and 6. It is a composite number.

Throughout the history of mathematics, people have looked for a formula that gives all the prime numbers. But no such formula has been found. The French mathematician Pierre Fermat stated that for any natural number n , $2^{(2^n)} + 1$ should always be a prime number. However, another mathematician, Euler, showed that this assumption was wrong by finding that $2^{(2^5)} + 1$ is divisible by 641.

Imagine that you want to find all the prime numbers which are less than a number n . The oldest and the most efficient way to do this was invented by the mathematician Eratosthenes (274-194 BC). It is called the **Sieve of Eratosthenes**. Using Eratosthenes' method, we make a list of all natural numbers less than or equal to n and cross out the multiples of all primes which are less than or equal to the square root of n . These numbers are not primes and the numbers that are left are the primes.

For example, look at the procedure for finding all primes less than or equal to 100:

List all numbers from 1 to 100.

The first prime number is 2. Cross out all multiples of 2 that are greater than 2 (4, 6, 8, etc.).

The next number left is 3. Cross out all multiples of 3 that are greater than 3.

The next number is 5. Cross out all multiples of 5 greater than 5.

The next number is 7. Cross out all multiples of 7 greater than 7.

The next number left, 11, is greater than the square root of 100.

So all of the remaining numbers are primes.

The table below shows the results of this process.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Goldbach's Conjecture

In a letter to Euler in 1742, the mathematician Goldbach wrote: 'Every odd integer greater than 9 is the sum of three odd primes'. Euler replied that this is equivalent to the following statement: 'Every even integer greater than 4 is the sum of two odd primes'.

For example,

$$6 = 3 + 3$$

$$8 = 3 + 5$$

$$10 = 3 + 7 = 5 + 5$$

$$12 = 5 + 7.$$

But this conjecture, known as Goldbach's conjecture, has not been proved yet.

Each of the circled numbers in the table is a prime number. The smallest prime number is 2.

We can write the prime numbers in a list:

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \dots\}.$$

Note that 2 is the only even prime number.

The difference between certain consecutive prime numbers is 2. These 'special' prime numbers are called **twin prime numbers**. For example, (3, 5), (5, 7), (11, 13) and (29, 31) are four examples of twin prime numbers. 43 and 47 are not twin primes because $47 - 43 > 2$.

Note

We can write any prime number except 2 and 3 in the form $6k + 1$ or $6k - 1$ ($k \in \mathbb{N}$). But this does not mean that every number of the form $6k + 1$ or $6k - 1$ ($k \in \mathbb{N}$) is a prime number.

The Sieve of Eratosthenes gives us a good way to find out whether or not a given number n is prime. All we need to do is divide n by all the primes that are less than its square root. If none of these numbers divide n then n is prime. For example, the square root of 211 is less than 15, so to verify that 211 is prime, we divide 211 by all the primes less than 15: 2, 3, 5, 7, 11 and 13. If none of these numbers divide 211 then 211 is prime. This way of determining whether or not a number is prime is sometimes called **trial division**.

EXAMPLE 16 Determine whether each number is prime or composite.

- a. 247 b. 313

Solution a. The square root of 247 is between 15 and 16 since $15^2 = 225$ and $16^2 = 256$. The prime numbers less than or equal to 16 are 2, 3, 5, 7, 11 and 13. Among these numbers, 13 divides 247. So 247 is not a prime number: it is composite.
 b. The square root of 313 is between 17 and 18 since $17^2 = 289$ and $18^2 = 324$. The prime numbers less than or equal to 18 are 2, 3, 5, 7, 11, 13 and 17. None of these numbers divides 313, so 313 is a prime number.

EXAMPLE 17 Explain why the sum of any two prime numbers other than 2 cannot be prime.

Solution Any prime number other than 2 is an odd number. The sum of two odd numbers is always even and greater than 2, so it cannot be prime.

EXAMPLE 18 Determine whether the number $M = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 3 \cdot 5 \cdot 7 \cdot 11 + 88 + 121$ is prime or not.

Solution By substituting $88 = 8 \cdot 11$ and $121 = 11 \cdot 11$, we can rewrite the given natural number as

$$\begin{aligned} M &= (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) + (3 \cdot 5 \cdot 7 \cdot 11) + 88 + 121 \\ &= (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) + (3 \cdot 5 \cdot 7 \cdot 11) + (8 \cdot 11) + (11 \cdot 11) \\ &= 11 \cdot (2 \cdot 3 \cdot 5 \cdot 7 + 3 \cdot 5 \cdot 7 + 8 + 11) \\ &= 11k, k \in \mathbb{N}. \end{aligned}$$

Since 11 is a factor of M , M is divisible by 11. It follows that M is not prime.

EXAMPLE 19 Determine whether the number 8999999 is prime or not.

Solution $8999999 = 9000000 - 1 = 3000^2 - 1 = (3000 - 1)(3000 + 1) = 2999 \cdot 3001$
 Since 8999999 has a factor different from 1 and itself, it is not prime.

Check Yourself 4

Determine whether each number is prime or not.

1. 383 2. 469 3. 2467

Answers

1. prime 2. not prime 3. prime



C. DIVISIBILITY

1. General Properties of Divisibility

Notation

If a is divisible by b we write $b|a$ and say ' b divides a '. If a is not divisible by b we write $b \nmid a$. Some books write $b : a$ to mean $b|a$.

GENERAL PROPERTIES OF DIVISIBILITY

Let $a, b, c, m, n \in \mathbb{N}$. Then the following properties hold.

1. $\frac{0}{a} = 0$ and $\frac{a}{0}$ is undefined.
2. 1 is the only number that divides every number and is not divisible by another number.
3. If $a|b$ and $b|c$ then $a|c$ (e.g. 6 divides 24 and 24 divides 72, so 6 divides 72).
4. If $a|b$ then $ma|mb$, and if $ma|mb$ then $a|b$ (e.g. 5 divides 25 so $5 \cdot 3$ divides $25 \cdot 3$. The converse is also true).
5. If $a|b$ and $c|d$ then $ac|bd$ (e.g. 3 divides 18 and 4 divides 24, so $3 \cdot 4$ divides $18 \cdot 24$).
6. If $a|b$ and $a|c$ then $a|(b+c)$ (e.g. 3 divides 9 and 3 divides 27, so 3 divides $9+27$).
7. If $a|b$ and $a|c$ then $a|(mb+nc)$ (e.g. 4 divides 12 and 4 divides 28, so 4 divides $12 \cdot 3 + 28 \cdot 7$).

Proof

1. • Since $0 = a \cdot 0$, zero is a multiple of any natural number a , and so a divides zero.
• If zero divides a then there exists $n \in \mathbb{N}$ such that $a = 0 \cdot n$. So $a = 0$.
But this is a contradiction, since a is a natural number and $0 \notin \mathbb{N}$. This means that zero can not divide a .
2. • Since $a = 1 \cdot a$, 1 is a multiple of a , i.e. 1 divides a .
• If $1 = a \cdot b$ then $a = 1$ and $b = 1$. So 1 is not divisible by another number.
3. If $a|b$ then there exists $p_1 \in \mathbb{N}$ such that $b = ap_1$ (by the definition of divisibility).
If $b|c$ then there exists $p_2 \in \mathbb{N}$ such that $c = bp_2$.
So $bc = ap_1bp_2$, which means $c = ap_1p_2$. Hence, c is a multiple of a , i.e. a divides c .
4. • If $a|b$ then there exists $p \in \mathbb{N}$ such that $b = ap$.
Multiplying both sides by m we get $mb = map$. So ma divides mb .
• If $ma|mb$ then there exists $p \in \mathbb{N}$ such that $mb = map$.
Cancelling both sides by m we get $b = ap$. So a divides b .



5. If $a|b$ then there exists $p_1 \in \mathbb{N}$ such that $b = ap_1$.
 If $c|d$ then there exists $p_2 \in \mathbb{N}$ such that $d = cp_2$.
 Multiplying the above equations side by side we get $bd = ap_1cp_2$.
 Substituting $p \in \mathbb{N}$ for p_1p_2 we obtain $bd = acp$.
 So ac divides bd .
6. If $a|b$ then there exists $p_1 \in \mathbb{N}$ such that $b = ap_1$.
 If $a|c$ then there exists $p_2 \in \mathbb{N}$ such that $c = ap_2$.
 Adding the above equations side by side we get $b + c = ap_1 + ap_2$.
 We can factor this as $b + c = a(p_1 + p_2)$, where $p_1 + p_2 \in \mathbb{N}$.
 So a divides $b + c$.
7. As previously, we can write $b = ap_1$ and $c = ap_2$ where $p_1, p_2 \in \mathbb{N}$.
 Multiplying both sides of $b = ap_1$ by m we get $mb = map_1$.
 Multiplying both sides of $c = ap_2$ by n we get $nc = nap_2$.
 Adding the above equations side by side we get $mb + nc = map_1 + nap_2$.
 Factoring the equality gives $mb + nc = a(mp_1 + np_2)$, where $mp_1 + np_2 \in \mathbb{N}$.
 So a divides $mb + nc$.

2. Divisibility Rules

We have seen that a natural number is divisible by another natural number if the remainder after division in $\mathbb{N}$ is zero. We can use regular division or a calculator to check whether one number is divisible by another number. However, we can also use some special properties of natural numbers to find out if one natural number is divisible by another natural number. We call these properties **divisibility rules**. In this section we will look at the divisibility rules for 2, 3, 4, 5, 8, 9, 10 and 11 and then generalize our results for some other natural numbers. We will prove these rules later in this chapter (see section 5: Modular Arithmetic).

Let us begin with the rule for divisibility by 2.

DIVISIBILITY BY 2

A natural number is divisible by 2 if it is even, i.e. if the last digit is 0, 2, 4, 6 or 8.

EXAMPLE 20 Determine whether each number is divisible by 2 or not.

- a. 3897 b. 87686

Solution

a. 3897 is not divisible by 2 since it is not even.
 b. 87686 is divisible by 2 since it is even.

DIVISIBILITY BY 3

A natural number is divisible by 3 if the sum of its digits is divisible by 3.

- EXAMPLE 21** Determine whether each number is divisible by 3 or not.
a. 82176 b. 1798615

Solution a. 82176 is divisible by 3 since $8 + 2 + 1 + 7 + 6 = 24$ is a multiple of 3.
b. 1798615 is not divisible by 3 since $1 + 7 + 9 + 8 + 6 + 1 + 5 = 37$ is not a multiple of 3.

Note

The remainder obtained when a natural number is divided by 3 is equal to the remainder obtained when the sum of its digits is divided by 3.

- EXAMPLE 22** What is the remainder when 780241 is divided by 3?

Solution $7 + 8 + 0 + 2 + 4 + 1 = 22$ has remainder 1 when divided by 3. Therefore the number 780241 also has remainder 1 when divided by 3.

DIVISIBILITY BY 4

A natural number is divisible by 4 if the number formed by its last two digits is divisible by 4.

- EXAMPLE 23** Determine whether each number is divisible by 4 or not.
a. 10725 b. 71600236

Solution a. The last two digits are 25. 10725 is not divisible by 4 since 25 is not divisible by 4.
b. 71600236 is divisible by 4 since 36 is divisible by 4.

Note

The remainder obtained when a natural number is divided by 4 is equal to the remainder obtained when the number formed by its last two digits is divided by 4.

- EXAMPLE 24** What is the remainder when 73263 is divided by 4?

Solution The last two digits are 63. Since 63 has remainder 3 when divided by 4, the number 73263 also has remainder 3 when divided by 4.



EXAMPLE

25

Find the possible values of the digit m such that

- $78m214$ is divisible by 3.
- $6238m$ is divisible by 4.
- $14287m4$ is divisible by 3 and not divisible by 4.

Solution

- If $78m214$ is divisible by 3 then the sum of its digits must be a multiple of 3.

So we can write $7 + 8 + m + 2 + 1 + 4 = 3k$ ($k \in \mathbb{N}$), i.e. $22 + m = 3k$.

Since $22 + m = 3k$ and $0 \leq m \leq 9$, let us consider the multiples of 3 which are greater than or equal to 22:

$$22 + m = 24; \quad m = 2$$

$$22 + m = 27; \quad m = 5$$

$$22 + m = 30; \quad m = 8$$

$$22 + m = 33; \quad m = 11 \text{ is not a solution since } m \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

So m can take the value 2, 5 or 8.

- If $6238m6$ is divisible by 4 then the number formed by last two digits must be a multiple of 4.

This means that the two-digit number $m6$ must be divisible by 4.

Among all the possible two-digit numbers $m6$, only 16, 36, 56, 76 and 96 are divisible by 4.

So m can take the value 1, 3, 5, 7 or 9.

- If $14287m4$ is divisible by 3 then we can use the same approach as in part a:

$$1 + 4 + 2 + 8 + 7 + m + 4 = 26 + m = 3k, \quad k \in \mathbb{N}$$

$$26 + m = 27; \quad m = 1$$

$$26 + m = 30; \quad m = 4$$

$$26 + m = 33; \quad m = 7.$$

So 1428714, 1428744 and 1428774 are divisible by 3.

Since we are looking for numbers that are divisible by 3 and not divisible by 4, we need to check the above numbers for divisibility by 4.

1428714 is not divisible by 4 since 14 is not divisible by 4.

1428744 is divisible by 4 since 44 is divisible by 4.

1428774 is not divisible by 4 since 74 is not divisible by 4.

Consequently, the possible values of m are 1 and 7.



DIVISIBILITY BY 5

A natural number is divisible by 5 if its last digit is 0 or 5.

- EXAMPLE 26** Determine whether each number is divisible by 5 or not.
a. 780 b. 28079 c. 381755

Solution a. 780 is divisible by 5 since its last digit is 0.
b. 28079 is not divisible by 5 since its last digit is neither zero nor 5.
c. 381755 is divisible by 5 since its last digit is 5.

Note

The remainder obtained when a natural number is divided by 5 is equal to the remainder obtained when the last digit of the number is divided by 5.

- EXAMPLE 27** Given that the four-digit number $203m$ is divisible by 2 and 3, what are the possible remainders when $203m$ is divided by 5?

Solution To find the possible values of the remainder we must first find the possible values of m .
If $203m$ is divisible by 2 then m can take the values 0, 2, 4, 6, 8.
If $203m$ is divisible by 3 then m can take the values 1, 4, 7 (check the sum of the digits).
Consequently, if $203m$ is divisible by both 2 and 3 then the only possible value of m is 4.
So the given number is 2034. Since the remainder of 2034 divided by 5 is equal to the remainder when 4 (the last digit) is divided by 5, the answer is 4.

DIVISIBILITY BY 8

A natural number is divisible by 8 if the number formed by its last three digits is divisible by 8.

- EXAMPLE 28** The four-digit number $23mn$ is divisible by 5 and 8. Find the possible values of this number.

Solution Since $23mn$ is divisible by 5, n is 0 or 5.
If $n = 0$ then $3m0$ must be divisible by 8. The possible values of m are 2 and 6.
If $n = 5$ then $3m5$ must be divisible by 8. But there is no such possible value for m .
So the possible values are 2320 or 2360.

Note

The remainder obtained when a natural number is divided by 8 is equal to the remainder obtained when the number formed by its last three digits is divided by 8.

DIVISIBILITY BY 9

A natural number is divisible by 9 if the sum of its digits is divisible by 9.

EXAMPLE 29

Is the number 144432 divisible by 9? Is it divisible by 8?

Solution Since $1 + 4 + 4 + 4 + 3 + 2 = 18$ is divisible by 9, 144432 is divisible by 9.
Since 432 is divisible by 8, 144432 is also divisible by 8.

Note

The remainder obtained when a natural number is divided by 9 is equal to the remainder obtained when the sum of its digits is divided by 9.

DIVISIBILITY BY 11

A natural number is divisible by 11 if the difference between the sum of its odd-numbered digits and the sum of its even-numbered digits is divisible by 11.

'Odd-numbered digit' means a digit that is in an odd place, i.e. the 1st, 3rd, 5th etc. digit in the number, from right to left. Similarly, 'even-numbered digit' means a digit in the 2nd, 4th or 6th etc. place in the number.

EXAMPLE 30

Determine whether each number is divisible by 11 or not.

- a. 1824 b. 908836742

Solution a. Counting from right to left, the sum of the odd-numbered digits is $4 + 8 = 12$.

The sum of the even-numbered digits is $2 + 1 = 3$.

The difference between these is $12 - 3 = 9$, which is not a multiple of 11. Consequently, 1824 is not divisible by 11.

- b. Again counting from right to left, the sum of the odd-numbered digits is $2 + 7 + 3 + 8 + 9 = 29$.

The sum of the even-numbered digits is $4 + 6 + 8 + 0 = 18$.

$29 - 18 = 11$ is a multiple of 11. Consequently, 908836742 is divisible by 11.

Note

The remainder obtained when a natural number is divided by 11 is equal to the remainder obtained when the difference between the sum of its odd-numbered digits and the sum of its even-numbered digits is divided by 11.

Check Yourself 5

- Find the remainder when 23048 is divided by 2, 3, 4, 5, 8, 9 and 11.
- The five-digit number 743ab is divisible by 5 and 9. Find the possible values of this number.

Answers

2. 74340, 74385



EXAMPLE 31

382ab is a five-digit even number which has remainder 1 when it is divided by 5 or 11. Find the possible values of a .

Solution Since $382ab \div 5$ has remainder 1, b is 1 or 6.

Since 382ab is even, b can be only 6.

On the other hand, if $382a6 \div 11$ has remainder 1, we can write

$$(6 + 2 + 3) - (a + 8) = 11k + 1, \quad k \in \mathbb{N}$$

$$3 - a = 11k + 1.$$

Since a is a digit, $0 \leq a \leq 9$. So the only possible value of k is 0. So $a = 2$.

REMAINDER OF THE SUM AND PRODUCT OF TWO NUMBERS

Let x and y be two natural numbers. If the remainder when x is divided by a is m , and the remainder when y is divided by a is n , then the following statements hold.

1. The remainder when $(x + y)$ is divided by a is equal to the remainder when $(m + n)$ is divided by a .
2. The remainder when xy is divided by a is equal to the remainder when mn is divided by a .

Proof If $x \div a$ has remainder m then $x = aq_1 + m$, where $q_1, m \in \mathbb{N}$.

If $y \div a$ has remainder n then $y = aq_2 + n$, where $q_2, n \in \mathbb{N}$.

$$1. \quad x + y = aq_1 + m + aq_2 + n = a(q_1 + q_2) + m + n$$

Since $a(q_1 + q_2)$ is divisible by a , the remainder of $(x + y) \div a$ is equal to the remainder of $(m + n) \div a$.

$$2. \quad xy = (aq_1 + m)(aq_2 + n) = a^2q_1q_2 + aq_1n + aq_2m + mn = a(aq_1q_2 + q_1n + q_2m) + mn$$

Since $a(aq_1q_2 + q_1n + q_2m)$ is divisible by a , the remainder of $xy \div a$ is equal to the remainder of $mn \div a$.

EXAMPLE 32

$A = 880543123$ and $B = 9062193$ are given. Find the remainder when $A \cdot B$ is divided by 9.

Solution Trying to find the product of such large numbers and then the remainder is not very practical. Instead we can use the property we have just learned.

When $A = 880543123$ is divided by 9 the remainder is 7 since the sum of its digits (34) has remainder 7 when divided by 9.

In a similar way we find that when $B = 9062193$ is divided by 9 the remainder is 3.

By the property, the remainder when $A \cdot B$ is divided by 9 is the same as the remainder when $7 \cdot 3 = 21$ is divided by 9. So the remainder is 3.



EXAMPLE

33

Find the remainder when $(2004 \cdot 10875) + 4640985 - 285$ is divided by 4.

Solution

2004 is divisible by 4, i.e. it has remainder 0 when divided by 4. We do not need to find the remainder when 10875 is divided by 4 (can you see why?). When 4640985 and 285 are divided by 4, they both have remainder 1. By using the above property we can now find the required remainder: $0 + 1 - 1 = 0$.

GENERAL RULE OF DIVISIBILITY (1)

Let p and q be two prime numbers and $c \in \mathbb{N}$. Then $p|c$ and $q|c$ if and only if $pq|c$.

Proof

We will prove that $p|c$ and $q|c$ implies that $pq|c$.

If $p|c$ then $c = pn_1$, $n_1 \in \mathbb{N}$. If $q|c$ then $c = qn_2$, $n_2 \in \mathbb{N}$. So $pn_1 = qn_2$, i.e. $n_1 = \frac{qn_2}{p}$.

Since p and q are prime numbers, p cannot divide q . So if $n_1 \in \mathbb{N}$ then p must divide n_2 : $\frac{n_2}{p} \in \mathbb{N}$.

This means $n_2 = pn_3$, $n_3 \in \mathbb{N}$. So $c = qn_2 = qpn_3$ which proves that c is divisible by pq .

The proof of the converse of this rule ($pq|c$ implies that $p|c$ and $q|c$) is left as an exercise for you.

With the help of the general rule of divisibility we can find the divisibility rules for different numbers. For example:

A natural number is divisible by 6 if it is divisible by both 2 and 3.

A natural number is divisible by 15 if it is divisible by both 3 and 5.

EXAMPLE

34

Find the possible ordered pairs (a, b) such that the five-digit number $80a1b$ is divisible by 15.

Solution

Since $80a1b$ is divisible by 15 then by the general rule of divisibility it must be divisible by the prime numbers 5 and 3. Hence the units digit is either 0 or 5, and the sum of the digits is a multiple of 3.

If $b = 0$ then $8 + 0 + a + 1 + 0 = 9 + a = 3k$, $k \in \mathbb{N}$.

$$9 + a = 9; a = 0$$

$$9 + a = 12; a = 3$$

$$9 + a = 15; a = 6$$

$$9 + a = 18; a = 9 \text{ are the possible values.}$$

If $b = 5$ then $8 + 0 + a + 1 + 5 = 14 + a = 3k$, $k \in \mathbb{N}$.

$$14 + a = 15; a = 1$$

$$14 + a = 18; a = 4$$

$$14 + a = 21; a = 7 \text{ are the possible values.}$$

So the possible pairs (a, b) are $(0, 0)$, $(3, 0)$, $(6, 0)$, $(9, 0)$, $(1, 5)$, $(4, 5)$ and $(7, 5)$.



EXAMPLE 35

The five-digit number $6m24n$ is divisible by 6. Find the greatest possible value of this number.

Solution Since $6m24n$ is divisible by 6, it must be divisible by both 2 and 3. So n must be even and $6 + m + 2 + 4 + n$ must be a multiple of 3.

$6m24n$ takes its greatest value when m and n are as big as possible. We can begin by making m as big as possible. 9 is the biggest digit and $m = 9$ implies $6m24n$ becomes $6924n$.

So $6 + 9 + 2 + 4 + n = 21 + n$ must be a multiple of 3, where n is even. $n = 6$ is the greatest possible digit that satisfies these two conditions. Consequently, the greatest possible value of the number is 69246.

EXAMPLE 36

How many elements of the set $\{1, 2, 3, \dots, 601\}$ are divisible by 4 or 7?

Solution Let us write the elements in consecutive order:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..., 16, ..., 599, 600, 601.

It is easy to see that every fourth number is divisible by 4. We can divide 601 by 4 to find how many such numbers there are (150).

$$\begin{array}{r} 601 \overline{) 4} \\ \underline{600} \\ 1 \end{array} \rightarrow \text{number of multiples of 4}$$

Similarly, to find how many elements are divisible by 7, we can divide 601 by 7 ($= 85$).

$$\begin{array}{r} 601 \overline{) 7} \\ \underline{56} \\ 41 \\ \underline{35} \\ 6 \end{array} \rightarrow \text{number of multiples of 7}$$

How many elements are divisible by 4 or 7? Notice that the answer is not $150 + 85$ since this sum includes numbers that are divisible by both 4 and 7 twice. In other words, we have counted numbers such as 28, 56, etc. twice: once as a multiple of 4 and once as a multiple of 7. Notice that these numbers will all be multiples of $4 \cdot 7 = 28$. There are $601 \div 28 = 21$ multiples of 28 in the set, so we should subtract this number from the sum.

$$\begin{array}{r} 601 \overline{) 28} \\ \underline{56} \\ 41 \\ \underline{28} \\ 13 \end{array} \rightarrow \text{number of multiples of 28}$$

Therefore, there are $150 + 85 - 21 = 214$ numbers between 1 and 601 that are divisible by 4 or 7.

Check Yourself 6

1. State the rule for divisibility by 21.
2. Find the remainder when $(2005 \cdot 2006) + 120675 - 98267232981$ is divided by 11.
3. Find the possible values of m if the four-digit number $35m0$ is divisible by 15.
4. Find the possible values of k if the four-digit number $7k26$ is divisible by 66.
5. How many elements of the set $\{1, 2, 3, \dots, 578\}$ are divisible by 3 or 5?

Answers

1. the number must be divisible by 3 and 7 2. 1 3. $m \in \{1, 4, 7\}$ 4. 3 5. 269



D. GREATEST COMMON FACTOR AND LEAST COMMON MULTIPLE

1. Prime Factorization

We have seen that any natural number greater than one is either prime or composite. If a number n is composite, it must have at least one divisor between 1 and n .

The divisors of n are called **factors** of n , and a representation of n by its factors is called a **factorization** of n . If m is the smallest divisor of n , m must be prime (otherwise it would be divisible by a number other than itself and 1, so it would not be the smallest divisor of n). We can use this argument to reduce all the divisors of n to prime numbers, and thus show that every composite number can be expressed as the product of two or more prime factors.

For example, $12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$, and both 2 and 3 are prime. 12 is factorized completely and the set of prime factors of 12 is $\{2, 3\}$.

Definition

prime factorization

The expression of a composite natural number as a product of primes is called the **prime factorization** of that number.

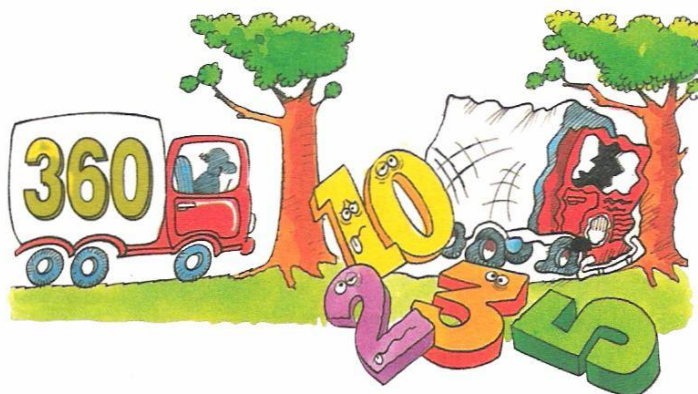
For example,

$$360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^3 \cdot 3^2 \cdot 5 \text{ and}$$

$$48510 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 7 \cdot 11 = 2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11$$

are prime factorizations of 360 and 48510.

There are no primes except 2, 3, 5, 7 and 11 that will divide 48510, even though it is such a large number. Of course, other numbers such as 6, 10, 14 and 33 are also factors of 48510, but in fact all these numbers reduce to products of the primes 2, 3, 5, 7 and 11.



Note

An important theorem in mathematics states that every composite natural number can be factorized into primes in only one unique way. This theorem is called the **Fundamental Theorem of Arithmetic**.

By the Fundamental Theorem of Arithmetic, $2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11$ is the *only* possible prime factorization of 48510; we cannot factorize 48510 in any other way using its prime factors.

EXAMPLE 37 Find the number whose prime factorization is $2^3 \cdot 3^2 \cdot 5 \cdot 7$.

Solution To find the number we just calculate the given product:

$$2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 8 \cdot 9 \cdot 35 = 2520.$$

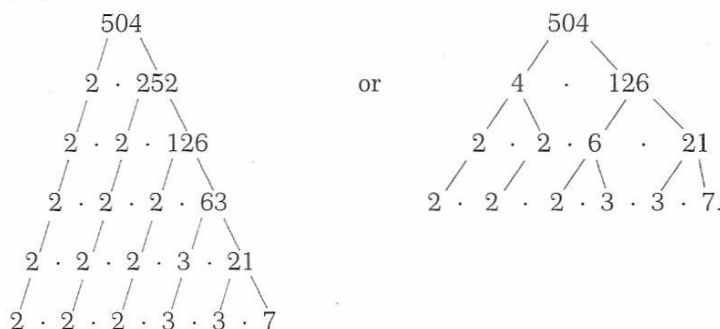
EXAMPLE 38 Find the prime factorization of 504.

Solution 1 The first few prime numbers are 2, 3, 5, 7, 11. One way to find the prime factorization of a number is to divide the number by the smallest possible prime factor each time, and stop when the quotient reaches 1. For example,

504	2	$504 \div 2 = 252$	(divide 504 by 2, quotient is 252)
252	2	$252 \div 2 = 126$	(divide 252 by 2, quotient is 126)
126	2	$126 \div 2 = 63$	(divide 126 by 2, quotient is 63)
63	3	$63 \div 3 = 21$	(63 is not divisible by 2, try 3, quotient is 21)
21	3	$21 \div 3 = 7$	(divide 21 by 3, quotient is 7)
7	7	$7 \div 7 = 1$	(7 is not divisible by 3, try 5, not divisible, try 7, quotient is 1)
1	stop	1.	(stop)

Now look at the prime factors in the column. We can write the prime factorization of 504 as $504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 = 2^3 \cdot 3^2 \cdot 7$. By the Fundamental Theorem of Arithmetic, we know that this is the only possible prime factorization.

Solution 2 We can also find the prime factorization of a number in another way: we choose any pair of factors (not necessarily prime factors) and split these factors until all the factors are prime. For example,



No matter how we split the factors, in the end we get the same answer:

$$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 = 2^3 \cdot 3^2 \cdot 7.$$



Theorem

Let M be a composite number whose prime factorization is $M = p_1^{a_1} \cdot p_2^{a_2} \cdot \dots \cdot p_n^{a_n}$. Then the following statements are true.

a. The number of factors of M is given by $S = (a_1 + 1)(a_2 + 1) \cdot \dots \cdot (a_n + 1)$.

b. The sum of the factors of M is given by

$$T = (1 + p_1 + p_1^2 + \dots + p_1^{a_1}) \cdot (1 + p_2 + p_2^2 + \dots + p_2^{a_2}) \cdot \dots \cdot (1 + p_n + p_n^2 + \dots + p_n^{a_n}).$$

c. The product of the factors of M is given by $P = M^{S/2}$, where S is the number of factors.

Proof

We will only prove part a of the theorem here.

We can rewrite $M = p_1^{a_1} \cdot p_2^{a_2} \cdot \dots \cdot p_n^{a_n}$ as $M = \underbrace{p_1 \cdot p_1 \cdot \dots \cdot p_1}_{a_1 \text{ times}} \underbrace{p_2 \cdot p_2 \cdot \dots \cdot p_2}_{a_2 \text{ times}} \dots \underbrace{p_n \cdot p_n \cdot \dots \cdot p_n}_{a_n \text{ times}}$.

First note that $p_1, p_2, \dots, p_n$ are prime factors of M , and so they are all different ($p_1 \neq p_2 \neq \dots \neq p_{n-1} \neq p_n$). Now note that

$1, p_1, p_1^2, \dots, p_1^{a_1}$ are factors of M ($a_1 + 1$ factors)

$1, p_2, p_2^2, \dots, p_2^{a_2}$ are factors of M ($a_2 + 1$ factors)

$\vdots$

$1, p_n, p_n^2, \dots, p_n^{a_n}$ are factors of M ($a_n + 1$ factors).

Again, because $p_1, p_2, \dots, p_n$ are prime and every different combination of primes is unique (by the Fundamental Theorem of Arithmetic), all of the factors we have counted above are unique.

In addition, we know that if any two primes p_x and p_y divide M then their product $p_x p_y$ also divides M . Consequently, there are $(a_1 + 1)(a_2 + 1) \cdot \dots \cdot (a_n + 1)$ divisors of M .

EXAMPLE

39

Find the number of factors of each number, and the sum and product of these factors.

a. 72

b. 300

Solution

We will use the theorem. Let S be the number of factors, T be the sum of the factors and P be their product.

a. The prime factorization of 72 is

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2, \text{ so}$$

$$S = (3 + 1) \cdot (2 + 1) = 12$$

$$T = (1 + 2 + 2^2 + 2^3) (1 + 3 + 3^2) = 195$$

$$P = 72^{S/2} = 72^{12/2} = 72^6.$$

$$b. 300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = 2^2 \cdot 3^1 \cdot 5^2$$

$$S = (2 + 1) \cdot (1 + 1) \cdot (2 + 1) = 18$$

$$T = (1 + 2^1 + 2^2) (1 + 3^1) (1 + 5^1 + 5^2) = 868$$

$$P = 300^{S/2} = 300^{18/2} = 300^9$$

Check Yourself 7

- Find the prime factorization of each number: 99000, 2006, 32190.
- Find the number of positive divisors of each number: 24, 60.
- A number $4 \cdot 10^n$ has 24 divisors. Find n .

Answers

- 99000 = $2^3 \cdot 3^2 \cdot 5^3 \cdot 11$, 2006 = $2 \cdot 17 \cdot 59$, 32190 = $2 \cdot 3 \cdot 5 \cdot 29 \cdot 37$
- 24: 8 divisors, 60: 12 divisors.
- 3

2. Greatest Common Factor

Definition

common factor

Let a and b be two natural numbers. If d is a factor of both a and b then d is called a **common factor** of a and b .

For example,

the set of all factors of 12 is $\{1, 2, 3, 4, 6, 12\}$ and

the set of all factors of 30 is $\{1, 2, 3, 5, 6, 10, 15, 30\}$.

So 1, 2, 3 and 6 are common factors of 12 and 30.

Definition

greatest common factor

The **greatest common factor** of two natural numbers a and b is the largest natural number that divides both a and b . It is denoted by $\text{GCF}(a, b)$, or sometimes just (a, b) .



The greatest common factor is also called the greatest common divisor, abbreviated to GCD. So $\text{GCF}(a, b) = \text{GCD}(a, b)$.

For example, the set of common factors of 12 and 30 is $\{1, 2, 3, 6\}$. So the greatest common factor of 12 and 30 is 6: $\text{GCF}(12, 30) = 6$.

Similarly, $\text{GCF}(24, 36) = 12$, $\text{GCF}(14, 34) = 2$ and $\text{GCF}(6, 25) = 1$.

We can find the greatest common factor of two or more numbers by using prime factorization. For example, let us find $\text{GCF}(936, 666)$. We begin by finding the prime factorizations:

936	2	666	2	
468	2	333	3	
234	2	111	3	
117	3	37	37	$936 = 2^3 \cdot 3^2 \cdot 13$
39	3	1		$666 = 2 \cdot 3^2 \cdot 37$
13	13			
1				

Now we find the common prime bases in each factorization and choose the smallest common exponent for each of these bases. The common bases are 2 and 3.

Choose 2: (the smallest exponent from 2 and 2^3).

Choose 3^2 : (the smallest exponent from 3^2 and 3^2).

The product of these numbers is $2 \cdot 3^2 = 18$. This is the greatest common factor of 936 and 666: $\text{GCF}(936, 666) = 18$.



The number 2 has exponent 1 since $2 = 2^1$.

EXAMPLE

40

Find $\text{GCF}(99, 102)$.

Solution

The prime factorizations are $99 = 3^2 \cdot 11$ and $102 = 2 \cdot 3 \cdot 17$. The only common base is 3 and its smallest exponent is 1. So $\text{GCF}(99, 102) = 3^1 = 3$.



EXAMPLE

41

The numbers 943 and 671 have remainders 7 and 5 respectively when they are divided by the natural number x . What is the greatest possible value of x ?

Solution

We can write $943 = xk + 7$ and $671 = xm + 5$ where $k, m \in \mathbb{N}$.

Simplifying gives us $936 = xk$ and $666 = xm$.

So $x \mid 936$ and $x \mid 666$. The greatest possible value of x is therefore equal to the greatest common factor of 936 and 666, which we can calculate using prime factorization, as before.

After calculation, $x = \text{GCF}(936, 666) = 18$.

An alternative way to find the greatest common factor of two or more numbers is as follows: Divide each of the numbers by a common prime factor, beginning with 2. If the common prime factor divides all the numbers then circle this common prime factor. If the prime factor does not divide all of the numbers, try the next prime number. Continue until no prime will divide all the quotients. The greatest common factor is the product of the circled common prime factors.

Look at the steps for finding the greatest common factor of 48, 72 and 120:

48	72	120	2	(2 divides all three numbers, so circle 2 and write the quotients)
24	36	60	2	(2 divides all three numbers, so circle 2 and write the quotients)
12	18	30	2	(2 divides all three numbers, so circle 2 and write the quotients)
6	9	15	2	(2 does not divide 9 or 15, do not circle but write the quotients)
3	9	15	3	(2 does not divide any number, try 3. 3 divides all numbers, so circle 3.)
1	3	5	3	(3 does not divide all numbers, so do not circle)
	1	5	5	(3 does not divide any number, so try 5. Do not circle.)
		1		$\text{GCF}(48, 72, 120) = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3 = 24$.

EXAMPLE

42

Find $\text{GCF}(60, 72, 84)$.

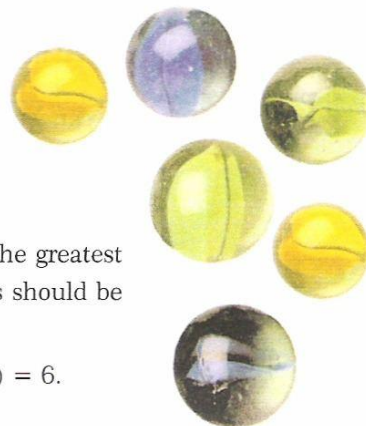
Solution

60	72	84	2	$\text{GCF}(60, 72, 84) = 2^2 \cdot 3 = 12$
30	36	42	2	
15	18	21	2	
15	9	21	3	
5	3	7	3	
5	1	7	5	
1		7	7	
		1		



EXAMPLE 43

A shopkeeper has 18 blue marbles, 30 yellow marbles and 36 green marbles to sell. He wants to put the marbles in different bags so that each bag contains the same number of marbles. In addition, marbles with different colors must not be in the same bag. At least how many bags does the shopkeeper need?



Solution For the least number of bags, the shopkeeper must put the greatest possible number of marbles in each bag, and no marbles should be left out.

So the number of marbles in each bag is $\text{GCF}(18, 30, 36) = 6$.

The total number of marbles is $18 + 30 + 36 = 84$.

As a result, the number of bags needed is $\frac{84}{6} = 14$.

Definition**relatively prime numbers**

Let a and b be two natural numbers. If $\text{GCF}(a, b) = 1$ then a and b are called **relatively prime numbers**.

For example, 12 and 35 are relatively prime since there is no natural number except 1 that divides both 12 and 35. In other words, $\text{GCF}(12, 35) = 1$.

Recall that in the previous part of this section we learned a general rule of divisibility based on prime numbers. By this rule, if a number is divisible by 33 then it must be divisible by 3 and 11, because 3 and 11 are prime factors of 33. Now we will extend this rule to relatively prime numbers.

GENERAL RULE OF DIVISIBILITY (2)

Let p and q be two relatively prime numbers and $c \in \mathbb{N}$. Then $p|c$ and $q|c$ if and only if $pq|c$.

For example, since 3 and 8 are relatively prime numbers, any number divisible by both 3 and 8 is also divisible by $3 \cdot 8 = 24$. Also, since 4 and 9 are relatively prime, any number divisible by both 4 and 9 is also divisible by 36. However, if a number is divisible by both 2 and 4 then it is not necessarily divisible by 8, since 2 and 4 are not relatively prime numbers.

EXAMPLE 44

Given that $n \in \mathbb{N}$, show that 100^n is divisible by 20.

Solution 100^n is divisible by 4 since 100 is divisible by 4 (00 is divisible by 4).

100^n is divisible by 5 since 100 is divisible by 5 (the last digit is 0).

Since $20 = 4 \cdot 5$ and 4 and 5 are relatively prime ($\text{GCF}(4, 5) = 1$) then by the general rule of divisibility, 100^n is divisible by 20.



EXAMPLE

45

Show that 1296 is divisible by 36.

Solution

We can begin by writing $36 = 4 \cdot 9$. Note that 4 and 9 are relatively prime since $\text{GCF}(4, 9) = 1$. By the general rule of divisibility, $4 \mid 1296$ and $9 \mid 1296$ implies that $36 \mid 1296$. So we need to show $4 \mid 1296$ and $9 \mid 1296$.

1296 is divisible by 9 since $1 + 2 + 9 + 6 = 18$ is a multiple of 9.

1296 is divisible by 4 since 96 is divisible by 4.

So 1296 is divisible by 36.

Is it true that if a number is divisible by 3 and 12 then it is divisible by 36? Why / Why not?

3. Least Common Multiple

In this section we will study the properties of the **least common multiple** of a set of numbers. The least common multiple is especially useful when we need to add or subtract fractions.

Definition

common multiple

A **common multiple** of two natural numbers is a number that is a multiple of both numbers.

For example,

the set of all multiples of 3 is $\{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, \dots\}$ and

the set of all multiples of 4 is $\{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, \dots\}$.

So $\{12, 24, 36, \dots\}$ is the set of common multiples of 3 and 4.

Definition

least common multiple

The **least common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b . It is denoted by $\text{LCM}(a, b)$, or sometimes just $[a, b]$.

For example, the set of common multiples of 3 and 4 is $\{12, 24, 36, \dots\}$. The least common multiple of 3 and 4 is 12: $\text{LCM}(3, 4) = 12$.

Similarly, $\text{LCM}(6, 8) = 24$, $\text{LCM}(15, 40) = 120$ and $\text{LCM}(6, 35) = 210$.

We can find the least common multiple of a set of numbers by using prime factorization. For example, let us find $\text{LCM}(270, 280, 300)$ in this way. We can calculate $270 = 2 \cdot 3^3 \cdot 5$, $280 = 2^3 \cdot 5 \cdot 7$ and $300 = 2^2 \cdot 3 \cdot 5^2$.

We begin by listing all the primes in the factorizations: 2, 3, 5 and 7. Then for each prime we choose the factor with the largest exponent in the set of factorizations:

2^3 (the largest exponent from 2 , 2^2 and 2)

3^3 (the largest exponent from 3 and 3^3)

5^2 (the largest exponent from 5 and 5^2)

7 (the largest exponent from 7).

The product of these numbers is $2^3 \cdot 3^3 \cdot 5^2 \cdot 7 = 37800$. This is the least common multiple of 270, 280 and 300: $\text{LCM}(270, 280, 300) = 37800$.



EXAMPLE 46 Find LCM(60, 72, 84).

Solution $60 = 2^2 \cdot 3 \cdot 5$

$$72 = 2^3 \cdot 3^2$$

$$84 = 2^2 \cdot 3 \cdot 7$$

$$\text{LCM}(60, 72, 84) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$$

An alternative way of finding the least common multiple of a set of numbers is as follows:

We begin by constructing a table as we did to find the greatest common factor. The only difference is that we do not circle any of the factors, and we multiply all of the numbers in the column to find the least common multiple.

For example, let us find the least common multiple of 48, 72 and 120:

48	72	120	2
24	36	60	2
12	18	30	2
6	9	15	2
3	9	15	3
1	3	5	3
	1	5	5
		1	

$$\text{LCM}(48, 72, 120) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 720.$$

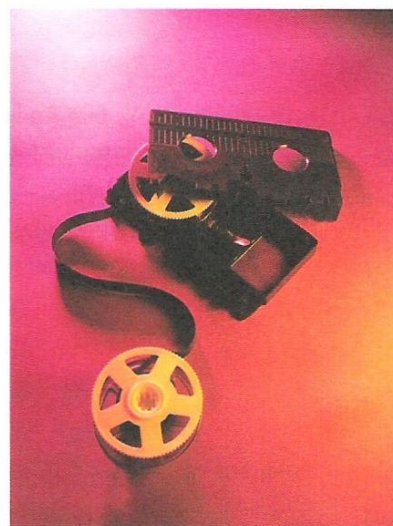
EXAMPLE 47 Two movie clubs show movies non-stop, and each club begins its first movie at 12:00. The movie shown in the first club lasts one hour and the movie shown in the second club lasts 100 minutes. When will the two movies begin again at the same time?

Solution First we convert one hour to 60 minutes. Now we need to find the smallest number which is divisible by 60 and 100. In other words, we need to find LCM(60, 100).

60	100	2
30	50	2
15	25	3
5	25	5
1	5	5
	1	

$$\text{LCM}(60, 100) = 2^2 \cdot 3 \cdot 5^2 = 300$$

This means that the movies will start together again after 300 minutes or five hours, i.e. at 17:00.



4. Mixed Problems

EXAMPLE 48 Find the product of $\text{GCF}(21, 35)$ and $\text{LCM}(21, 35)$.

Solution

$$\begin{array}{r|l} 21 & 35 \\ 7 & 35 \\ 7 & 7 \\ 1 & 1 \end{array}$$

$$\text{GCF}(21, 35) = 7$$

$$\text{LCM}(21, 35) = 3 \cdot 5 \cdot 7$$

$$\text{GCF}(21, 35) \cdot \text{LCM}(21, 35) = 3 \cdot 5 \cdot 7^2 = 3 \cdot 7 \cdot 5 \cdot 7 = 735$$

Note that $735 = 21 \cdot 35$.

Theorem

Let a and b be two natural numbers. Then $\text{GCF}(a, b) \cdot \text{LCM}(a, b) = ab$.

Proof

Let a and b be two natural numbers with the following prime factorizations:

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \quad \text{and} \quad b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}.$$

We find the common prime bases in each factorization and choose the smallest common exponent for each prime. The product of these numbers is the greatest common factor of a and b .

$$\text{GCF}(a, b) = p_1^{\min(a_1, \beta_1)} p_2^{\min(a_2, \beta_2)} \cdots p_n^{\min(a_n, \beta_n)} \quad (1)$$

Now we list the primes belonging to either of the factorizations, and choose the largest exponent in the set of factorizations. The product of these numbers is the least common multiple of a and b .

$$\text{LCM}(a, b) = p_1^{\max(a_1, \beta_1)} p_2^{\max(a_2, \beta_2)} \cdots p_n^{\max(a_n, \beta_n)} \quad (2)$$

Multiplying equations (1) and (2) side by side gives

$$\text{GCF}(a, b) \cdot \text{LCM}(a, b) = p_1^{a_1 + \beta_1} p_2^{a_2 + \beta_2} \cdots p_n^{a_n + \beta_n} = ab.$$

EXAMPLE 49 Given $\text{GCF}(a, b) = 20$ and $\text{LCM}(a, b) = 420$, find all the possible pairs (a, b) which satisfy $a > b$.

Solution

If $\text{GCF}(a, b) = 20$ then 20 is the largest number dividing both a and b .

Hence we can write $a = 20m$ and $b = 20n$ ($m, n \in \mathbb{N}$).

$$\text{Also, } \text{GCF}(a, b) \cdot \text{LCD}(a, b) = 20 \cdot 420$$

$$ab = 8400$$

$$20m \cdot 20n = 8400$$

$$mn = 21.$$

This suggests that $m = 7$ and $n = 3$ or $m = 21$ and $n = 1$ (since $a > b$).

Substituting these values into the expressions $a = 20m$ and $b = 20n$ gives

$$a = 20 \cdot 7 = 140 \quad \text{and} \quad b = 20 \cdot 3 = 60, \text{ or}$$

$$a = 20 \cdot 21 = 420 \quad \text{and} \quad b = 20 \cdot 1 = 20.$$

Consequently, the possible ordered pairs (a, b) are $(140, 60)$ and $(420, 20)$.

EXAMPLE 50

When a number is divided by 7 the remainder is 5, when it is divided by 5 the remainder is 3, and when it is divided by 3 the remainder is 1. Find the smallest possible value of this number.

Solution Let the number be A , so $A - 5$, $A - 3$ and $A - 1$ are divisible by 7, 5 and 3, respectively. We can rewrite this as: $A + 2 - 7$, $A + 2 - 5$ and $A + 2 - 3$ are divisible by 7, 5 and 3, respectively. So $A + 2$ is divisible by 7, 5 and 3.

For the smallest possible value, $A + 2 = \text{LCM}(7, 5, 3) = 105$. So the number is 103.

Another good way to find the greatest common factor of two numbers is by using a process called the **Euclidean algorithm**.

For example, let us find $\text{GCF}(936, 666)$ using the Euclidean algorithm:

$$\begin{array}{ll} 936 = 666 \cdot 1 + 270 & \text{(divide 936 by 666, find the remainder)} \\ 666 = 270 \cdot 2 + 126 & \text{(divide 666 by 270, find the remainder)} \\ 270 = 126 \cdot 2 + 18 & \text{(divide 270 by 126, find the remainder)} \\ 126 = 18 \cdot 7 + 0 & \text{(divide 126 by 18, find the remainder)} \end{array}$$

We stop when the remainder is zero. The last remainder before zero is the greatest common factor: $\text{GCF}(936, 666) = 18$.

We can also calculate the least common multiple of two numbers from their greatest common factor, using the theorem which states that $\text{GCF}(a, b) \cdot \text{LCM}(a, b) = ab$.

For example, using $\text{GCF}(936, 666) = 18$, we can find $\text{LCM}(936, 666)$ as follows:

$$\text{GCF}(936, 666) \cdot \text{LCM}(936, 666) = 936 \cdot 666, \text{ i.e. } 18 \cdot \text{LCM}(936, 666) = 936 \cdot 666.$$

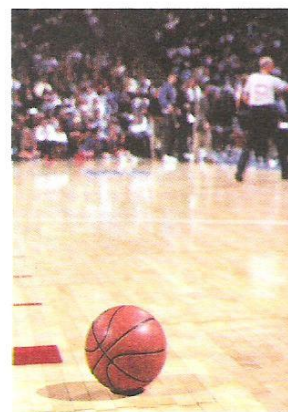
So $\text{LCM}(936, 666) = 34632$.

Check Yourself 8

- Find the greatest common factor and least common multiple of each set of numbers.
a. 20 and 30 b. 248, 304 and 500
- A number has remainder 5 when it is divided by 6, remainder 7 when it is divided by 8, and remainder 8 when it is divided by 9. Find the smallest possible value of this number.
- Twenty football players, sixteen volleyball players and twelve basketball players will stay in a hotel during a sports tournament. Each hotel room must have the same number of players, and only players from the same team can share a room. What is the minimum number of hotel rooms necessary for the players?
- Use the Euclidean algorithm to find $\text{GCF}(1232, 135)$.

Answers

1. a. $\text{GCF} = 10$, $\text{LCM} = 60$ b. $\text{GCF} = 4$, $\text{LCM} = 1178000$ 2. 71 3. 12 4. 1

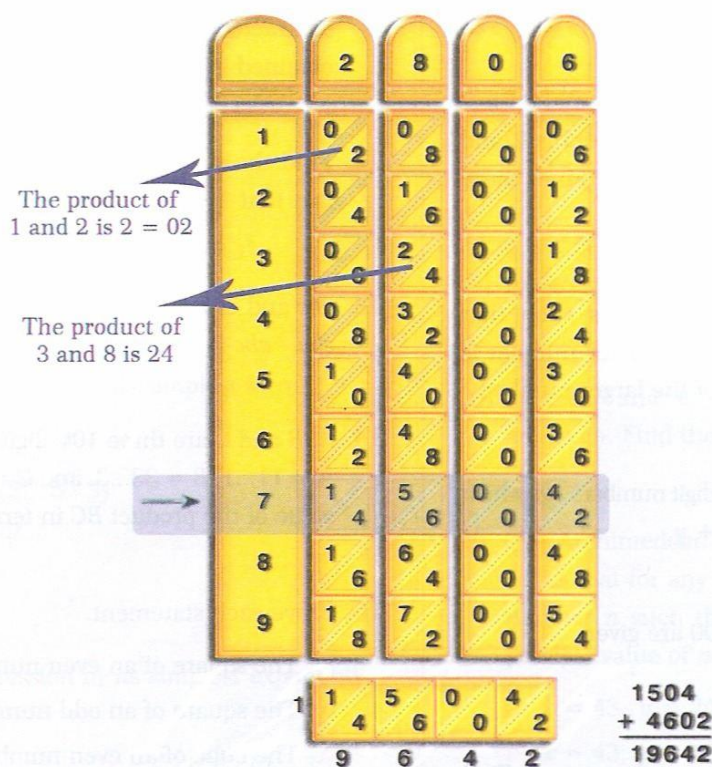


NAPIER'S RODS

The Scottish mathematician John Napier (1550-1617) introduced an important calculating tool called Napier's rods, or Napier's bones.

Napier's rods were a set of strips, several for each digit 0 through 9, on which the multiples of each digit appeared in two columns.

The figure below shows 7×2806 , calculated using Napier's rods. The result, 19642, is shown at the bottom of the figure.



Think about how to calculate the product 57×2806 using Napier's rods. Investigate how Napier's rods can be used to calculate the sum, difference and quotient of two natural numbers.

EXERCISES 1.1

A. Natural Numbers

1. Determine whether each of the following is a natural number or not.
 - a. $[20 - (3 \cdot 5)] \div 7$
 - b. $(26 - 7) \cdot 0$
 - c. $30 \div (6 \div 2)$
 - d. $[(36 - 12) - 4] \div 4 + 5$
2. Find three consecutive natural numbers whose sum is equal to the product of the first two numbers.
3. The difference of two natural numbers is 2 and the sum of their squares is 100. Find these numbers.
4. Find two consecutive odd numbers such that three times the square of the smaller number is two more than the square of the larger number.
5. Find the sum of all the three-digit numbers xyz which satisfy $x = 2 \cdot y$ and $z = x + y$.
6. $a + b = 10$ and $b + c = 100$ are given. Find $a + 3b + 2c$.
7. $a + b = 7$, $b + c = 8$ and $a + c = 9$ are given. Find the value of a .
8. $a, b, c \in \mathbb{N}$ and $ab = 8$, $a + b = 9$, $a - b = c$ are given. Find c .
9. Find the values of the two digits m and n if $6mn = mn6 + 9$, where $6mn$ and $mn6$ are three-digit numbers.
10. When the digits of a two-digit number are reversed, the reversed number is three times the sum of its digits. Find the original number.
11. The ones and hundreds digits of a three-digit number are interchanged and the new number is subtracted from the first number. The difference obtained is the three-digit number $59x$. Find x .
12. Find the sum $446 + 447 + \dots + 460$ using the fact that $1 + 2 + 3 + \dots + 14 = 105$.
13. abc and cba are two three-digit numbers such that $cba - abc = 198$. Find the maximum value of $a + c$.
14. A , B and C are three 100-digit numbers such that $A = 11\dots 1$, $B = 33\dots 3$, and $C = 66\dots 6$. What is the value of the product BC in terms of A ?
15. Prove each statement.
 - a. The square of an even number is also even.
 - b. The square of an odd number is also odd.
 - c. The cube of an even number is also even.
 - d. The cube of an odd number is also odd.
16. Evaluate the expressions.
 - a. 6^3
 - b. 2^7
 - c. 10^5
 - d. 82^1
 - e. 3^5





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17. If possible, write the expression as an exponential expression.

- a. $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$
 b. $a \cdot a \cdot a \cdot a \cdot a$
 c. $6 + 6 + 6 + 6 + 6 + 6$
 d. $2 + 2 + 2 + 2$
 e. $\underbrace{a^2 + a^2 + a^2 + \dots + a^2}_{a \text{ times}}$
 f. $\underbrace{b + b + b + \dots + b}_{b^5 \text{ times}}$

18. Write each expression as an expression with a single exponent.

- a. $a^3 \cdot a^3$ b. $2^m \cdot 2^{2m} \cdot 2^{3m}$
 c. $x^2 \cdot x^3 \cdot x^6$ d. $(m^2)^6$
 e. $(n^4)^5$ f. $(3^5)^{12}$
 g. $(x^3)^7 \cdot x^5$ h. $(a^7)^3 \cdot (a^6)^2$
 i. $81^a \cdot 729^{4a}$

19. Write each expression in its simplest form.

- a. $(2^3 \cdot 5^4) + (2^5 \cdot 5^4)$
 b. $(2^4 \cdot 5^3 \cdot 27)(2^2 \cdot 5 \cdot 3)$
 c. $[(5^2 \cdot 3^4)^2 \cdot 4^3]^2$
 d. $3^5 - 3^4$

20. Write each expression in its simplest exponential form.

- a. $729^3 \cdot 81^5$ b. $(2^4 \cdot 6^2)^3 \cdot (2^7 \cdot 3^2)^4$

21. $3^a = m$ is given. Write each expression in terms of m .

- a. 9^a b. 27^{a+1} c. 81^{2a+5} d. 243^{3a+1}

22. Write each expression in its simplest form.

- a. $(a^3b^4)(a^5b^3)$
 b. $(x^2y^3)^2 x^6y^4$
 c. $a^4(a^2b^3)^2 b^3$
 d. $(2m^2n^3)^3 (m^2n)^2$
 e. $(4a^3)^5 (16a)^2$
 f. $[(x^2)^5 y]^2 (xy^2xy^3)^3$
 g. $(9a^2bc^2)^2 (4ab^3c)^4 (6a^3bc^4)^3$
 h. $(15xy^2z^2)^2 (75x^2yx^3)^3$

23. How many zeros does each number have at the end?

- a. $128 \cdot 5^7$ b. $3^2 \cdot 40^5 \cdot 600^7$
 c. $(150^3 \cdot 600)^4$ d. $12 \cdot 11 \cdot 10 \cdot \dots \cdot 2 \cdot 1$

24. $a, b \in \mathbb{N}$ and $a = 37 - x^5$, $b = x^5 - 7$ are given. Find the greatest possible value of the product ab .

25. $m, n \in \mathbb{N}$ and $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 4^n \cdot m$ are given. Find the greatest possible value of n .

26. The Archimedean property of natural numbers states that for any $x, y \in \mathbb{N}$, there exists a natural number n such that $nx > y$. Find the smallest possible value of n in each case.

- a. $x = 43$, $y = 24$
 b. $x = 43$, $y = 1924$
 c. $x = 58$, $y = 4827$

27. $a, b, c \in \mathbb{N}$ and $ab = 8$, $bc = 10$, $b > 1$ are given. Find $2a + b - c$.



28. Find two natural numbers a and b which satisfy each pair of conditions.

- a. $103 = 6a + b$, $0 \leq b < 6$
 b. $103 = 6a + b$, $6 \leq b < 12$
 c. $103 = 6a + b$, $24 \leq b < 30$
 d. $103 = 6a + b$, $42 \leq b < 48$

29. Show that any natural number that is the square of another natural number has remainder zero or 1 when it is divided by 4.

B. Prime Numbers

30. Determine whether each of the following is a prime number or not.

- a. 383 b. 389
 c. 1549 d. 2467
 e. 71253 f. $7^{25} - 13$
 g. $5! + 7!$ note: $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$
 h. $2^7 - 1$

31. $a, b, c \in \mathbb{N}$, $\frac{a+1}{c} = \frac{c}{b+2}$ and c is a prime number.

Order a , b and c .

32. p , q and r are different prime numbers such that $p + q = r$ and $r < 100$. Find the minimum and maximum possible value of r .

33. How many prime numbers p satisfy the condition $(17! + 1) < p < (17! + 19)$?

34. $\begin{array}{r} a \overline{) b} \\ \underline{} \\ b-1 \end{array}$ The division on the left is correct for any prime number b that is less than 8. Find the smallest possible value of a .

C. Divisibility

35. Test each number for divisibility by 2, 3, 4, 5, 8, 9 and 11 using the divisibility rules.

- a. 3826 b. 86 c. 47983
 d. 695 e. 7816 f. 986867332

36. Let A be a set such that $A = \{1, 2, 3, 4, \dots, 578\}$.

- a. How many elements of A are divisible by 3?
 b. How many elements of A are divisible by 5?
 c. How many elements of A are divisible by both 3 and 5?
 d. How many elements of A are divisible by 3 or 5?
 e. How many elements of A are divisible by 3, but not divisible by 5?

37. When a number A is divided by 7, the quotient is B and the remainder is 4. When B is divided by 3, the quotient is C and the remainder is 2. Find the remainder when A is divided by 21.

38. $635ab$ is a five-digit number such that $a - b = 4$. Find the sum of the possible values of a which make $635ab$ a multiple of 4.

39. When the five-digit number $213ab$ is divided by 10, the remainder is 2. The number is also divisible by 3 and 4. Find the possible values of a .

40. A five-digit number $x435y$ is divisible by 11. Find the sum of the possible values of x and y .

41. A five-digit number $4a53b$ is divisible by both 5 and 9. Find the possible values of a .

42. A two-digit number ab is divisible by 9. Find the remainder when $7a4b2$ is divided by 9.

43. A five-digit number $23a4b$ is divisible by 6. What is the greatest possible value of $a + b$?



44. Find the remainder when each number is divided by 9.

a. $\underbrace{474747\dots47}_{108 \text{ digits}}$

b. $\underbrace{66666\dots6}_{20 \text{ digits}}\underbrace{8888\dots8}_{20 \text{ digits}}$

45. $x = 4y + 5$ and $y = 6z + 4$ are given. Find the remainder when x is divided by 12.

46. Find the number of divisors of each number.

a. 400 b. 2000 c. 3125 d. $12 \cdot 11 \cdot \dots \cdot 1$

47. m , n and p are prime numbers such that $240 = m^a \cdot n^b \cdot p^c$. Find the value of $a + b + c$.

48. The number of divisors of $16 \cdot 25^n$ is 75. Find n .

49. Find the sum of all the divisors of each number.

a. 24 b. 60 c. 144 d. 240

50. a. How many divisors does 18000 have?

b. Find the sum of these divisors.

c. Find the product of these divisors.

51. How many divisors of $22^2 + 44^2 + 66^2$ are a multiple of 7?

D. Greatest Common Factor and Least Common Multiple

52. Find the greatest common factor of each set of numbers.

a. 18 and 30 b. 99 and 275
c. 130 and 455 d. 12, 18 and 30
e. 252, 308 and 504 f. 4500, 15000 and 4320

53. Find the least common multiple of each set of numbers.

a. 120 and 320 b. 28 and 70
c. 56 and 96 d. 24, 36 and 48
e. 24, 54 and 72 f. 15, 45 and 120

54. Find the value of the digit m if the four-digit number $35m4$ is divisible by 36.

55. $A = 32m0n$ is a five-digit number with $n < 5$. Find the number of possible values of m which make A divisible by 12.

56. Determine whether the following numbers are relatively prime or not.

a. 70 and 120 b. 124 and 85
c. 72 and 84 d. 16, 25 and 49
e. 22, 35 and 117

57. A and B are two natural numbers such that $\text{GCF}(A, B) = 2^2 \cdot 3$ and $\text{LCM}(A, B) = 2^2 \cdot 3^2 \cdot 5$. Find the product AB .

58. a is a natural number such that
✱ $\text{GCF}(a, 90, 225) = 15$ and $\text{LCM}(a, 90, 225) = 1350$. Find the smallest possible value of a .

59. $3m$ and $4n$ are two two-digit numbers such that $\text{GCF}(3m, 4n) = 6$ and $\text{LCM}(3m, 4n) = 252$. Find $m + n$.

60. a. Find the smallest natural number which has remainder 8 when it is divided by 12, 14 and 18.
b. Find the smallest natural number which has remainder 5 when it is divided by 6, 7 and 8.

61. Find the smallest natural number which has remainder 5 when it is divided by 6, remainder 7 when it is divided by 8, and remainder 8 when it is divided by 9.



62. Find the smallest natural number which has remainders 5, 4 and 3 when it divides 17, 22 and 33, respectively.
63. Let $x, y, z \in \mathbb{N}$. Find the smallest value of x which satisfies $7x + 5 = 8y + 6 = 9z + 7$.
64. Find the smallest value of x which satisfies $x = 5a = 6b + 2 = 7c$, where a, b and $c \in \mathbb{N}$.
65. Find all natural numbers x and y such that $\text{GCF}(x, y) = 8$ and $\text{LCM}(x, y) = 80$.
66. Use the Euclidean algorithm to find the greatest common factor of each pair of numbers.
- 396 and 210
 - 3691 and 628
 - 1586 and 1112
 - 7392 and 1356
 - 13020 and 4564
67. A rectangular field is 18 m wide and 24 m long. A man wants to divide the field into congruent square regions. At least how many square regions will be inside the field?
68. Fernando and Kimi are in a karting race, following a circular track. If they start at the same place and drive in the same direction, Fernando completes one revolution every two minutes and Kimi completes one revolution every 80 seconds. How long will it take them to meet at the starting point again?



Mixed Problems

69. Find the remainder when $1 \cdot 2 \cdot 3 \cdot \dots \cdot 1995 - 1$ is divided by $100 \cdot 101$.
70. Prove that the product of any two consecutive even numbers is divisible by 8.
71. Let p be a prime number such that $p > 3$. Prove that $p^2 - 1$ is divisible by 24.
72. Find the possible values of $n \in \mathbb{N}$ which make $\frac{3n+25}{n+3} - 2n$ a natural number.
73. A *perfect square* is a number that is the square of a natural number. n is a natural number with three factors such that $2n + 1$ is a perfect square. Find all possible values of n .
74. Let $x, y \in \mathbb{N}$ such that $x > y$ and $x^2 - y^2$ is a prime number. Show that $x^2 - y^2 = 2y + 1$.
75. A man born in the nineteenth century was x years old in the year x^2 . Find the year of his birth.
76. ab and ba are two two-digit numbers. Find the value of $a^2 + b^2$ if $(ab)^2 - (ba)^2 = 1089$.
77. Let n be any odd number. Show that $n^2 - 1$ is divisible by 8.
78. a and b ($b \neq 0$) are two digits such that the eight-digit number $aaaaa2ab$ is divisible by 24. Find all possible values of a and b .
79. a, b and c are different natural numbers such that $a + b = 7$ and $b - 1 = c^2$. Find the possible values of a .

2

INTEGERS

A. POSITIVE AND NEGATIVE INTEGERS

So far we have discussed the set of natural numbers $\{1, 2, 3, 4, \dots\}$. Clearly these numbers alone are not enough for all the number problems in everyday life. For example, natural numbers cannot be used to write some winter temperatures, since such temperatures may be less than zero.

To have a wider set of numbers we introduce a minus (i.e. negative) sign for numbers less than zero. Numbers with a negative sign are called **negative numbers**. The union of the set of negative numbers with the set of natural numbers and zero gives us a new number set: the set of **integers**.

Definition

integer

The set of **integers** is denoted by $\mathbb{Z}$:

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}.$$

We write $\mathbb{Z}^- = \{\dots, -4, -3, -2, -1\}$ to mean the set of **negative integers** and $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$ to mean the set of **positive integers**. The set of all positive integers together with zero is called the set of **non-negative integers**, and the set of all negative integers together with zero is called the set of **non-positive integers**.

INTEGERS AROUND US

The tip of this iceberg is 47 m above sea level.

This diver is swimming at -25 m.

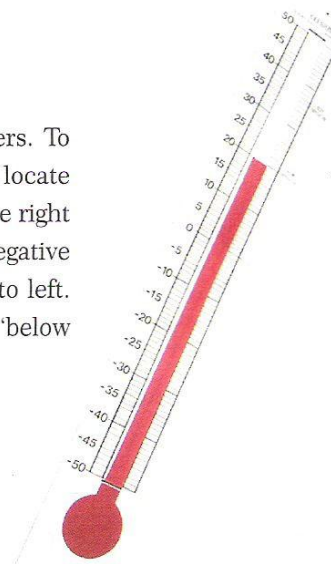
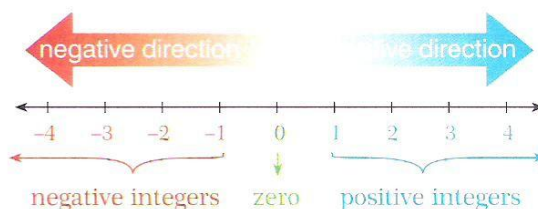
Water freezes at 0°C .

The letter Z comes from the German word *zahlen*, which means 'to count'.

Note

1. $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$
2. $\mathbb{Z}^+ = \mathbb{N}$
3. Zero is neither negative nor positive.

A number line provides a good picture of the set of integers. To draw a number line, we begin with a horizontal line and locate zero on it. We write the positive integers 1, 2, 3, 4, etc. to the right of zero, equally spaced from left to right. We write the negative numbers -1, -2, -3, -4, etc. to the left of zero, from right to left. We can see that a number line is like a thermometer, with 'below zero' meaning 'to the left of zero'.



We say that an integer a is less than an integer b if a is to the left of b on the number line. For example, -2 is less than 4.

B. OPERATIONS ON INTEGERS

1. Addition and Subtraction

The rules for adding integers are given below.

- The sum of two positive integers is a positive integer (e.g. $12 + 5 = 17$).
- The sum of two negative integers is a negative integer (e.g. $-4 + (-14) = -18$).
- The sum of two integers with opposite signs may be a positive or negative integer or zero (e.g. $-21 + 32 = 11$, $17 + (-26) = -9$ and $23 + (-23) = 0$).

The numbers 9 and -9 are not the same: a temperature of 9°C above zero is not the same as a temperature of 9°C below zero! However, 9 and -9 share one common feature: they are the same distance from zero on the number line. Because of this, their sum is zero. Any two numbers whose sum is zero are called **additive inverses** of each other. For example, 17 is the additive inverse of -17, and vice versa. The additive inverse of zero is also zero.

We can now say that subtraction is a form of addition, since the difference of two integers is the first integer plus the additive inverse of the second integer.

For example, $3 - 8 = 3 + (-8) = -5$ and $-5 - (-7) = -5 + (+7) = 2$.



EXAMPLE**51**

m and n are two integers which satisfy $-1 \leq m < 5$, $-7 \leq n \leq 2$. Find the maximum possible value of each expression.

a. $m + n$

b. $m - n$

Solution

a. At the maximum value of $m + n$, m must take its maximum value and n must also take its maximum value. So $m + n = 5 + 2 = 7$.

b. At the maximum value of $m - n$, m must take its maximum value and n must take its minimum value. So $m - n = 5 - (-7) = 12$.

2. Multiplication and Powers

The rules for multiplying integers are given below.

- The product of two positive integers is a positive integer (e.g. $24 \cdot 3 = 72$).
- The product of two negative integers is also a positive integer (e.g. $(-7) \cdot (-9) = 63$).
- The product of two integers with opposite signs is a negative integer (e.g. $8 \cdot (-6) = -48$).

We can use these rules to determine the sign of a power of an integer:

- Any natural number power of a positive integer is a positive integer.
- Any even power of a negative integer is a positive integer.
- Any odd power of a negative integer is a negative integer.

For example, $(-2)^3 = -(2^3) = -8$, $(-3)^6 = 3^6 = 729$ and $(2)^3 = 2^3 = 8$.

We must be careful when we write and evaluate powers of negative numbers.

For example, $(-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$ and $-(2^3) = -(2 \cdot 2 \cdot 2) = -8$. So $(-2)^3 = -(2^3)$. But $(-2)^4 \neq -(2^4)$, because $(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 16$ and $-(2^4) = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$.

EXAMPLE**52**

a , b and c are integers such that $a < 0$, $b > 0$ and $c < 0$. Determine whether the product a^3b^2c is positive or negative.

Solution

If $a < 0$ then $a^3 < 0$. If $b > 0$ then $b^2 > 0$. So a^3b^2 is negative $((-) \cdot (+) = (-))$.

We know that $c < 0$. So the product a^3b^2c is positive $((-) \cdot (-) = (+))$.

EXAMPLE**53**

Evaluate each expression without using a calculator.

a. $(-13) \cdot 997$

b. $[2005^2 - (8020 \cdot 1003) + (-2006)^2]^5$

Solution

$$\text{a. } (-13) \cdot 997 = -13 \cdot (1000 - 3) = -(13 \cdot 1000 - 13 \cdot 3) = -13000 + 39 = -12961$$

$$\begin{aligned} \text{b. } [2005^2 - (8020 \cdot 1003) + (-2006)^2]^5 &= (2005^2 - (2 \cdot 2005 \cdot 2 \cdot 1003) + 2006^2)^5 \\ &= (2005^2 - 2 \cdot 2005 \cdot 2006 + 2006^2)^5 \\ &= ((2005 - 2006)^2)^5 = ((-1)^2)^5 = 1 \end{aligned}$$

EXAMPLE 54 $x, y, z \in \mathbb{Z}^-$ and $5x = 8y = 3z$ are given. Write x, y and z in increasing order.

Solution Let $5x = 8y = 3z = A$. Then $A < 0$, since x, y and z are negative integers.
If $5x = 8y$ then x is less than y since $5x$ and $8y$ have the same negative product.
If $8y = 3z$ then y is greater than z since $8y$ and $3z$ have the same negative product.
If $5x = 3z$ then x is greater than z since $5x$ and $3z$ have the same negative product.
Combining these results gives $z < x < y$.

3. Division and Mixed Operations

A division without remainder is the inverse operation of multiplication, so the sign of the result of division is the same as the sign for multiplication. In other words, the division of two integers with the same sign gives a positive result and the division of two integers with opposite signs gives a negative result.

For example, $(-21) \div 7 = -3$, $48 \div (-1) = -48$ and $(-15) \div (-3) = 5$.

Since division with a remainder has a non-integer result, we will study it later.

EXAMPLE 55 Evaluate $\frac{24 \div 12 \div 2}{(-6) \div (-3) \div (-2)}$.

Solution $\frac{24 \div 12 \div 2}{(-6) \div (-3) \div (-2)} = \frac{2 \div 2}{2 \div (-2)} = \frac{1}{-1} = -1$

EXAMPLE 56 x and y are integers such that $y = 2 - \frac{9}{x+1}$. Find the sum of all the possible values of x .

Solution If y is an integer then $\frac{9}{x+1}$ is also an integer since 2 is an integer. So $x + 1$ must divide 9. All the possible values of $x + 1$ are therefore $-9, -3, -1, 1, 3, 9$. So the possible values of x are $-10, -4, -2, 0, 2, 8$. Their sum is -6 .

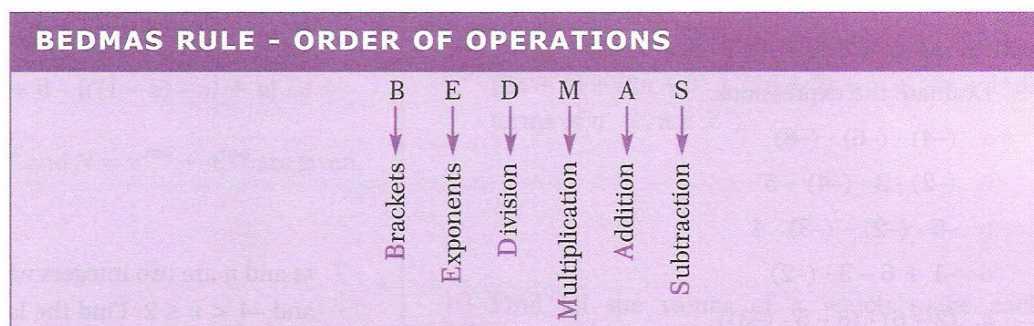
EXAMPLE 57 Given that $\frac{4a+18}{a}$ is an integer, find all possible values of a .

Solution Let $\frac{4a+18}{a} = k$, where k is an integer. Then we have $4a + 18 = ak$, i.e. $(k - 4)a = 18$. Since $k - 4$ is an integer, a is also an integer and must divide 18 without remainder. So the possible values of a are $-18, -9, -6, -3, -2, -1, 1, 2, 3, 6, 9, 18$.

Expressions such as $5 + 2 - 3$, $6 \cdot 4 - 7$ and $3 \cdot 5 - 6 \div 2$ involve two or more operations. We say that these expressions involve **mixed operations**. Sometimes we need to be careful when performing mixed operations: $(6 \cdot 4) - 7$ is not the same as $6 \cdot (4 - 7)$. The order in which we perform the operations becomes important. Therefore, we use a standard order for mixed operations:

1. Perform operations inside the brackets.
2. Simplify any expression with exponents.
3. Multiply or divide from left to right.
4. Add or subtract from left to right.

This operation order is also known as the **BEDMAS rule**, which takes its name from the first letters of the operations, in order.



EXAMPLE 58 Evaluate the expressions.

- a. $[16 \div 4 + 7 + (-2)] \div (-6 \div 2) + (-4) \cdot (5) + (-10)$
- b. $12 \div [(-1)^4 + 3] - [5^3 - (-1)^5] \div 2$

Solution We use the BEDMAS rule to determine the order of the operations.

- a. $[16 \div 4 + 7 + (-2)] \div (-6 \div 2) + (-4) \cdot (5) + (-10)$
 $= [4 + 7 - 2] \div (-3) + (-20) - 10 = 9 \div (-3) - 20 - 10 = -3 - 20 - 10 = -33$
- b. $12 \div [(-1)^4 + 3] - [5^3 - (-1)^5] \div 2$
 $= 12 \div [1 + 3] - [125 - (-1)] \div 2 = 12 \div 4 - 126 \div 2 = 3 - 63 = -60$

Check Yourself 9

1. $-2 \leq m < 4$ and $m \in \mathbb{Z}$ are given. Find the maximum possible value of $m^2 - 3m + 4$.
2. Evaluate $3 - (3 - (3 - (3 - 2)))$.
3. Evaluate $-[43 + (-14)][(-42) \div (-16 + 2) \div (-3)]$.
4. Evaluate $1 - 2 + 3 - 4 + \dots - 100$.

Answers

1. 14 2. 2 3. 29 4. -50



EXERCISES 1.2

B. Operations on Integers

1. Evaluate the expressions.

- $(8 - 5) - ((3 - 5) - 6)$
- $(-1) - (1 - 2 - (3 - 4))$
- $[-3 + (-5)] + [-(3 + 5)]$
- $10 - [9 - [6 - (7 - 4)] + 5]$
- $3 - 6 + 9 - 12 + 15 - 18$

2. Evaluate the expressions.

- $(-4) \cdot (-6) \cdot (-8)$
- $(-2) \cdot 3 \cdot (-4) - 5$
- $-5 \cdot (-2) - (-3) \cdot 4$
- $-1 + 6 - 3 \cdot (-2)$
- $(2 - 6) \cdot (8 - 5 \cdot (-3))$

3. Evaluate the expressions.

- $(-4)^2 - (-2)^3 - 4^2$
- $-3^2 - (-3)^3 - 2 \cdot (-3)$
- $(2^3)^2 \cdot (-2)^2 \div 2^6$
- $(10^{10} - 10^9) \div (10^9 - 10^8)$
- $2^{11} \cdot 3^9 + 3^9 \cdot 2^{10} - 6^{10}$

4. Evaluate the expressions.

- $(11 - 22) - (44 - 66) \cdot [72 - 7 \cdot 5 \cdot 2]$
- $[(-21 - 7) \div -4] \cdot 2$
- $[8 - (-4)] \div [3 \cdot (6 - 4)] - [5 - (-8)] \cdot [-6 - (-9)]$
- $[-(-74 + 32) \div (21 \div 7) + 4 \cdot (-8 + 7)] - [-(-6 + 23)]$
- $-[43 + (-14)] \cdot [(16 - 2) \div (-3 + 10)] + [-3 - (-42)]$

5. Evaluate the expressions without using a calculator.

- $(111 + 222 + 333 + 444 + 555) \div 111 \cdot 21$
- $(282000 + 28200 + 2820) \div (28200 + 2820 + 282)$

6. Given that $a, b, c \in \mathbb{Z}$, simplify the expressions.

- $a - [-(b - c) + 2(a - c) + 2(b - a)]$
- $[a + (c - (a - 1))] - b + 1 - [a - (b - (c + 2))]$

7. m and n are two integers which satisfy $-3 \leq m < 7$ and $-4 < n \leq 2$. Find the largest possible value of each expression.

- $3m + n$
- $m - n$
- $-2m + n$
- $m^2 + n^2$

8. For the same values of m and n given in question 7, find the smallest possible value of each expression.

- $2m - n$
- $4m + n$
- $m^2 - n^2$
- $-m + 3n$

9. x, y and z are three positive integers such that $xy = 14$, $x^2y = 28$ and $y^2z = 98$. Find $x + y + z$.



Mixed Problems

10. Evaluate

$$(1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \dots 4371 - 4372 + 4373).$$

11. Evaluate

$$(1 + 3 + 5 + \dots + 205) - (2 + 4 + 6 + \dots + 204).$$

12. $M = 5 \cdot 4^{1994} + 4^{1995}$ and $N = 2^{3989} + 4^{1996}$ are given.

Evaluate $\frac{N}{M}$.

13. a , b and c are three integers which satisfy $-5 < a \leq 3$, $-8 < b < 2$ and $b^2 - 2a + 3c = 6$. Find the minimum possible value of c .

14. x and y are two integers such that $2 < \frac{x}{y} < 6$ and $x + y = 9$. Find the maximum possible value of x .

15. a , b and c are three non-negative integers such that $a + b = 12$ and $b + c = 10$. Find the maximum possible value of $a + b + c$.

16. Find all possible values of $x, y \in \mathbb{Z}^+$ such that $4x + 5y = 35$.

17. Prove that there are no $a, b \in \mathbb{Z}$ which satisfy the equation $2a - 12b = 21$.

18. $1 + 3 + 5 + 7 + \dots + (2k - 1) = k^2$ is given. Find $(2n + 1) + (2n + 3) + (2n + 5) + \dots + (4n + 1)$ in terms of n . ($k, n \in \mathbb{Z}^+$)

19. Find all the values of a which make each expression an integer.

a. $\frac{5a^3 + 18a + 16}{a^2}$ b. $\frac{a^2 + 5a - 3}{a + 2}$
c. $\frac{a^2 + 5a - 6}{a^2 + 2}$

20. An integer is called *palindromic* if it reads the same backward as forward. For example, 44, 2552 and 3456543 are palindromic numbers.

- How many two-digit palindromic numbers are there?
- How many n -digit palindromic numbers are there?
- Prove that any palindromic four-digit number is divisible by 11.

3

RATIONAL NUMBERS

A. RATIONAL NUMBERS

So far we have seen how natural numbers can be used to count objects, and we have extended the set of natural numbers to the set of integers. But integers cannot describe every real-life situation. For example, we may eat one quarter of a pie, a camera may cost \$320.99, or a movie may last two and a half hours. To describe these situations, we need fractions, more formally called **rational numbers**. We cannot list the rational numbers as easily as we list the natural numbers and the integers. However, we can define the set of rational numbers formally as follows:



Definition

rational number, fraction, numerator, denominator

A number which can be expressed as the ratio of an integer to a non-zero integer is called a **rational number**. The set of rational numbers is denoted by $\mathbb{Q}$, i.e.

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

The number $\frac{a}{b}$ is called a **fraction**. In this fraction, a is the **numerator** and b is the **denominator**.

Note that every rational number can be written as a quotient of integers in an unlimited number of ways.



The letter $\mathbb{Q}$ comes from the word 'quotient'.

$\frac{1}{2}, \frac{5}{3}, \frac{-37}{8}, \frac{0}{17}, -\frac{8}{3}, \frac{12345}{54321}$ and $\frac{14}{1}$ are rational numbers and fractions. Some of these rational numbers can be expressed more simply. For example, $\frac{0}{17} = 0$ and $\frac{14}{1} = 14$. Notice that 14 is both an integer and a rational number. In fact, all integers are rational numbers since we can write any integer as a fraction with denominator 1.

Note

1. For all $a \in \mathbb{Z}$ and $a \neq 0$, $\frac{0}{a} \in \mathbb{Q}$ but $\frac{a}{0} \notin \mathbb{Q}$.
2. For all $a \in \mathbb{Z}$, $\frac{a}{1} = a \in \mathbb{Q}$. So $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$.
3. The set of rational numbers includes both negative and positive fractions.

If we divide the numerator of a fraction by its denominator then we can convert a rational number to **decimal** form.

For example, $\frac{7}{2} = 3.5$, $\frac{5}{3} = 1.666\dots$, $\frac{1}{8} = 0.125$ and $\frac{534}{100} = 5.34$.



People use a comma (,) in Europe or a period (.) in other countries to show the decimal point. In this book we use a period as the decimal point.

In the decimal number $a.b$, a is called the **whole part**, b is the **decimal part** and the period (.) is the **decimal point**. As a decimal number, every rational number either **repeats** (e.g. $1.666\dots$) or **terminates** (e.g. 3.5).

To convert a terminating decimal number to a fraction, we multiply it by the fraction $\frac{10^n}{10^n}$, where n is the number of digits in the decimal part.

For example, $2.17 = 2.17 \cdot \frac{10^2}{10^2} = \frac{217}{100}$, $-0.9 = -\frac{9}{10}$ and $12.487 = \frac{12487}{1000}$.

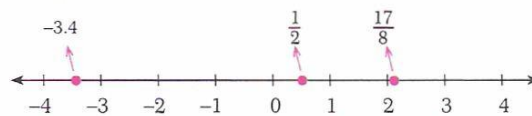
We will look at how to convert repeating decimals to fractions later in this section.

Every rational number corresponds to a point on the number line. For example,

$\frac{1}{2} = 0.5$ is half the distance between 0 and 1,

-3.4 is $\frac{4}{10}$ of the distance between -3 and -4 (working left from -3), and

$\frac{17}{8} = 2.125$ is $\frac{125}{1000}$ of the distance between 2 and 3 (working right from 2).

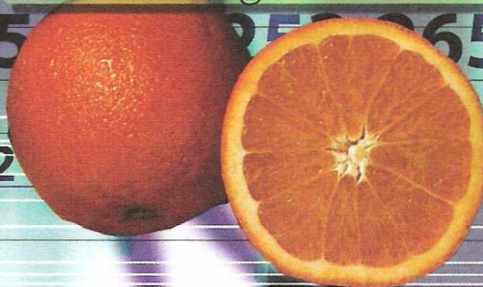


Rational Numbers around Us

Each slice is $\frac{1}{5}$ of the pizza.



There are one and a half oranges.



The frequency of the radio station is 96.8 FM.



B. FRACTIONS

1. Types of Fraction

There are three types of fraction:

- A **proper fraction** (also called a **simple fraction**) is a fraction in which the numerator is less than the denominator. $\frac{1}{2}$, $\frac{2}{5}$ and $\frac{5}{9}$ are proper fractions.
- An **improper fraction** is a fraction in which the numerator is equal to or larger than the denominator. $\frac{2}{1}$, $\frac{5}{3}$, $\frac{9}{5}$ and $\frac{2}{2}$ are improper fractions.
- A **mixed fraction** is a fraction consisting of a whole number and a proper fraction. $2\frac{1}{3}$, $4\frac{3}{5}$ and $7\frac{19}{20}$ are mixed fractions.

To convert an improper fraction to a mixed fraction, we divide the numerator by the denominator, and then write the whole part of this quotient as the whole part of the mixed fraction and the remainder as the new numerator. The denominator does not change.

For example, $\frac{7}{3} = 2\frac{1}{3}$ since $\frac{7}{3} = \frac{6}{3} + \frac{1}{3}$. Similarly, $\frac{129}{10} = 12\frac{9}{10}$ and $-\frac{11}{2} = -5\frac{1}{2}$.

To convert a mixed fraction to an improper fraction, we multiply the whole part by the denominator and add this product to the numerator. The denominator does not change.

For example, $3\frac{2}{5} = \frac{17}{5}$ since $(3 \cdot 5) + 2 = 17$. Similarly, $-4\frac{10}{13} = -\frac{62}{13}$ and $20\frac{3}{4} = \frac{83}{4}$.

EXAMPLE 59

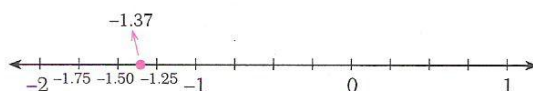
Write the decimal number -1.37 as a mixed fraction and show it on the number line.

Solution Since -1.37 has two digits in its fraction part we multiply it by $\frac{100}{100}$ to convert it to a fraction:

$-1.37 = -\frac{137}{100}$. This is an improper fraction, and we need to convert it to a mixed fraction.

Applying the division algorithm to $\frac{137}{100}$ gives us $\frac{137}{100} = \frac{100}{37} + \frac{37}{100}$. So $-\frac{137}{100} = -1\frac{37}{100}$.

The number is between -1 and -2 . To show it on the number line, we divide the number line into 100 parts between -1 and -2 . Then we move along 37 of these 100 parts in the negative direction, starting from -1 .



2. Equivalent Fractions

$\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. For example, $\frac{2}{5} = \frac{6}{15}$ since $2 \cdot 15 = 5 \cdot 6$.

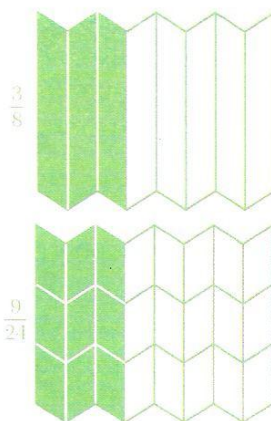
$\frac{2}{5}$ and $\frac{6}{15}$ are called **equivalent fractions** because they have the same value.

When we multiply the numerator and the denominator of a fraction by the same number we obtain a new fraction which is equivalent to the original one. We call this operation **building up**.

For example, $\frac{3}{8} = \frac{3 \cdot 3}{8 \cdot 3} = \frac{9}{24}$.

When we divide the numerator and the denominator of a fraction by the same number we have a new fraction which is equivalent to the original one. We call this operation **reducing**.

For example, $\frac{30}{36} = \frac{30 \div 6}{36 \div 6} = \frac{5}{6}$.



Building up or reducing a fraction does not change its value.

EXAMPLE 60 Find the fraction $\frac{x}{y}$ which is equivalent to $\frac{2}{5}$ such that $x + y = 21$.

Solution Since $\frac{x}{y} = \frac{2}{5}$ we have $5x = 2y$. (1)

Since $x + y = 21$ we have $y = 21 - x$. (2)

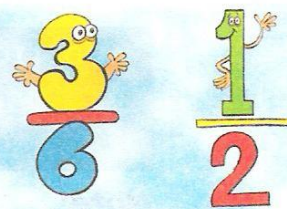
Substituting (2) in (1) we get $x = 6$ and $y = 15$. So the required fraction is $\frac{6}{15}$.

A fraction is called **reduced**, **simplified** or **in lowest terms** if its numerator and denominator are relatively prime, i.e. if their greatest common divisor is 1. When we perform operations on fractions, it is usually best to give the result in lowest terms.

EXAMPLE 61 Reduce $\frac{148}{370}$ to its lowest terms.

Solution Since $\text{GCD}(148, 370) = 74$,

we have $\frac{148}{370} = \frac{148 \div 74}{370 \div 74} = \frac{2}{5}$.



- How did you get such a beautiful figure?
- By reducing!

Check Yourself 10

1. Convert each fraction to a decimal and show it on the number line.

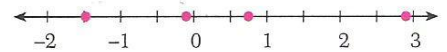
a. $\frac{3}{4}$ b. $-\frac{3}{20}$ c. $\frac{42}{15}$ d. $-1\frac{1}{2}$

2. Convert each decimal to a fraction and show it on the number line.

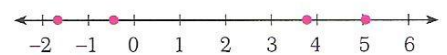
a. -0.45 b. 3.875 c. 5.04 d. -1.7

Answers

1. a. 0.75 b. -0.15 c. 2.8 d. -1.5



2. a. $-\frac{9}{20}$ b. $\frac{31}{8}$ c. $\frac{126}{25}$ d. $-\frac{17}{10}$



3. Adding and Subtracting Fractions

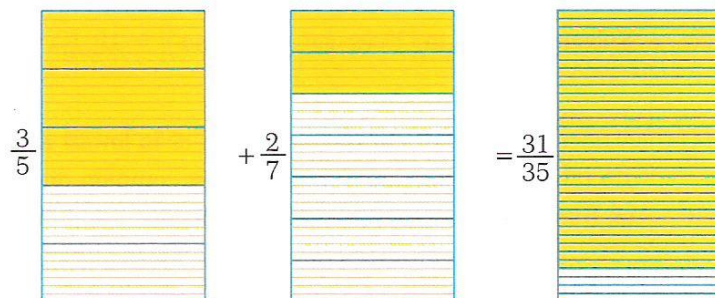
To add or subtract fractions which have the same denominator, we add or subtract the numerators and keep the common denominator. If the denominators are not the same then we find the least common multiple of the denominators and build up each fraction so that its denominator is the least common multiple.

For example, let us add $\frac{3}{5} + \frac{2}{7}$. The least common multiple of denominators 5 and 7 is 35.

Therefore we build up each fraction to have 35 as a denominator:

$$\frac{3}{5} + \frac{2}{7} = \frac{3 \cdot 7}{5 \cdot 7} + \frac{2 \cdot 5}{7 \cdot 5} = \frac{3 \cdot 7 + 2 \cdot 5}{35} = \frac{21 + 10}{35} = \frac{31}{35}$$

We can show this operation as follows:



ADDITION AND SUBTRACTION OF FRACTIONS

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

Note

$$a \frac{b}{c} = a + \frac{b}{c} = \frac{ac + b}{c}$$



EXAMPLE**62**

Perform the operations.

a. $\frac{7}{8} - \frac{5}{6}$

b. $5 + \frac{-7}{6}$

c. $\frac{4}{9} - 3\frac{2}{3}$

Solution

a. $\frac{7}{8} - \frac{5}{6} = \frac{7 \cdot 6 - 5 \cdot 8}{8 \cdot 6} = \frac{42 - 40}{48} = \frac{2}{48} = \frac{1}{24}$

b. $5 + \frac{-7}{6} = \frac{5}{1} + \frac{-7}{6} = \frac{5 \cdot 6 - 7 \cdot 1}{1 \cdot 6} = \frac{30 - 7}{6} = \frac{23}{6} = 3\frac{5}{6}$

c. $\frac{4}{9} - 3\frac{2}{3} = \frac{4}{9} - \frac{11}{3} = \frac{4 \cdot 3 - 11 \cdot 9}{9 \cdot 3} = \frac{-87}{27} = -\frac{29}{9}$ or $-3\frac{2}{9}$

EXAMPLE**63**Evaluate $\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{12} + 1$.**Solution**

The least common multiple of all the denominators is 12, so we build up each fraction to have denominator 12.

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{12} + 1 = \frac{6}{12} - \frac{3}{12} + \frac{4}{12} - \frac{1}{12} + \frac{12}{12} = \frac{6 - 3 + 4 - 1 + 12}{12} = \frac{18}{12} = \frac{3}{2}$$

EXAMPLE**64**Evaluate $\frac{23}{24} + \frac{24}{25} + \frac{25}{26} + \frac{26}{27}$ in terms of a if $a = \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27}$.**Solution**

$$\frac{23}{24} + \frac{24}{25} + \frac{25}{26} + \frac{26}{27} = (1 - \frac{1}{24}) + (1 - \frac{1}{25}) + (1 - \frac{1}{26}) + (1 - \frac{1}{27})$$

$$= 1 + 1 + 1 + 1 - (\frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27}) = 4 - a$$

Check Yourself 11

Perform the operations.

1. $\frac{7}{12} - \frac{5}{18}$

2. $3 - \frac{2}{3} + \frac{5}{12} - \frac{1}{8} + \frac{13}{16}$

3. $16 - 2\frac{2}{3} + \frac{15}{22} + 5 - \frac{16}{33} + \frac{9}{10}$

Answers

1. $\frac{11}{36}$

2. $\frac{55}{16}$

3. $\frac{3206}{165}$

4. Multiplying and Dividing Fractions

To multiply fractions we multiply the numerators to get the new numerator and denominators to get the new denominator.

For example, $\frac{3}{5} \cdot \frac{2}{7} = \frac{3 \cdot 2}{5 \cdot 7} = \frac{6}{35}$.

If we swap the places of the numerator and denominator in a fraction, the new number is called the **reciprocal** of the fraction. The product of a fraction and its reciprocal is always 1. The reciprocal of a number is also called its **multiplicative inverse**.

For example, the reciprocal of $\frac{2}{9}$ is $\frac{9}{2}$, and $\frac{2}{9} \cdot \frac{9}{2} = \frac{18}{18} = 1$.

To divide two rational numbers, we multiply the dividend by the reciprocal of the divisor.

For example, $\frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \cdot \frac{7}{2} = \frac{21}{10}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

EXAMPLE

65

Perform the operations.

a. $3\frac{1}{3} \cdot (-\frac{7}{4})$

b. $\frac{14}{9} \div 5$

c. $(-\frac{57}{2}) \div 5\frac{7}{10}$

Solution

a. $3\frac{1}{3} \cdot (-\frac{7}{4}) = \frac{10}{3} \cdot (-\frac{7}{4}) = -\frac{10 \cdot 7}{3 \cdot 4} = -\frac{70}{12} = -\frac{35}{6}$

b. $\frac{14}{9} \div 5 = \frac{14}{9} \div \frac{5}{1} = \frac{14}{9} \cdot \frac{1}{5} = \frac{14}{45}$

c. $(-\frac{57}{2}) \div 5\frac{7}{10} = (-\frac{57}{2}) \div \frac{57}{10} = (-\frac{57}{2}) \cdot \frac{10}{57} = -\frac{57 \cdot 10}{2 \cdot 57} = -5$

EXAMPLE

66

Evaluate $(3 - \frac{2}{5}) \div (\frac{5}{6} - \frac{1}{9}) \div \frac{12}{5} \cdot (-\frac{2}{3})$.

Solution

We can use the BEDMAS rule to determine the order of the operations.

$$(3 - \frac{2}{5}) \div (\frac{5}{6} - \frac{1}{9}) \div \frac{12}{5} \cdot (-\frac{2}{3}) = \frac{13}{5} \div \frac{13}{18} \div \frac{12}{5} \cdot (-\frac{2}{3}) = \frac{13}{5} \cdot \frac{18}{13} \cdot \frac{5}{12} \cdot (-\frac{2}{3}) = -1.$$



Rule

$$\text{GCF}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{GCF}(ad, bc)}{\text{LCM}(b, d)} \quad \text{and} \quad \text{LCM}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{LCM}(a, c)}{\text{GCF}(b, d)}$$

EXAMPLE 70

Find the greatest common factor and the least common multiple of $\frac{9}{4}$ and $\frac{12}{5}$.

Solution Using the given rule,

$$\text{GCF}\left(\frac{9}{4}, \frac{12}{5}\right) = \frac{\text{GCF}(9 \cdot 5, 12 \cdot 4)}{\text{LCM}(4, 5)} = \frac{3}{20} \quad \text{and} \quad \text{LCM}\left(\frac{9}{4}, \frac{12}{5}\right) = \frac{\text{LCM}(9, 12)}{\text{GCF}(4, 5)} = \frac{36}{1} = 36.$$

$\frac{0}{0}$ is meaningless!

Division by zero is impossible in any of the number sets. To understand why, first remember that $0 \cdot a = 0$. Also, $a \div b = c$ means $a = b \cdot c$.

Now suppose that division by zero is possible. In other words, suppose $a \div 0 = b$. We should then have $a = 0 \cdot b$, and thus $a = 0$. This means $a \div 0 = b$ is impossible if $a \neq 0$. So now suppose $0 \div 0 = b$. This gives $0 = 0 \cdot b$. But this is true for any value of b .

This means that $0 \div 0$ could be any number. However, the result of an arithmetic operation is equal to a unique value. For this reason, mathematicians have agreed that $0 \div 0$ is meaningless.

Check Yourself 12

Evaluate the expressions.

$$1. \quad 5 + \frac{5}{2} + 5 + \frac{5}{2}$$

$$2. \quad 2 - \frac{3 + \frac{1}{3}}{1 - \frac{2}{3}} \div \frac{\frac{5}{2} + \frac{3}{2}}{2 - \frac{1}{4}}$$

$$3. \quad 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{\ddots}}}}$$

Answers

1. 8 2. $-\frac{64}{59}$ 3. 1



5. Comparing and Ordering Fractions

We have seen how to compare integers and decimals on a number line: if a number a is to the left of another number b on the number line, then a is smaller than b . However, it is not always so easy to use a number line for ordering fractions if a fraction has a large or awkward denominator.

Remember that another way of comparing numbers is to consider their difference. If their difference is positive then the first number is the biggest; if their difference is negative then the first number is the smallest.

For example, $\frac{5}{13} < \frac{3}{7}$ because $\frac{5}{13} - \frac{3}{7} = -\frac{4}{91} < 0$.

EXAMPLE 71 Compare the fractions.

a. $\frac{4}{5}$ and $\frac{2}{3}$

b. $-\frac{4}{3}$ and $-\frac{5}{4}$

Solution a. $\frac{4}{5} - \frac{2}{3} = \frac{12-10}{15} > 0$. So $\frac{4}{5} > \frac{2}{3}$. b. $(-\frac{4}{3}) - (-\frac{5}{4}) = -\frac{4}{3} + \frac{5}{4} = \frac{-16+15}{12} < 0$. So $-\frac{4}{3} < -\frac{5}{4}$.

The proper fractions $\frac{3}{4}$, $\frac{6}{7}$ and $\frac{9}{10}$ share a common difference (1) between their numerator and denominator. Similarly, the improper fractions $\frac{5}{3}$, $\frac{9}{7}$ and $\frac{15}{13}$ have a common difference of 2 between their numerator and denominator. We can compare fractions such as these easily with a simple rule: as the values of numerator and the denominator increase,

1. the value of a proper fraction increases.
2. the value of an improper fraction decreases.

For example, $\frac{2}{5} < \frac{3}{6} < \frac{4}{7} < \frac{5}{8} < \dots$ and $\frac{7}{2} > \frac{8}{3} > \frac{9}{4} > \frac{10}{5} > \frac{11}{6} > \dots$.

EXAMPLE 72 How many integers satisfy the inequality $\frac{1}{5} < \frac{d}{30} < \frac{1}{3}$?

Solution Let us build up each fraction so that each denominator is the least common multiple of the denominators. The least common multiple is $\text{LCM}(5, 30, 3) = 30$.

Then the inequality $\frac{1}{5} < \frac{d}{30} < \frac{1}{3}$ becomes $\frac{6}{30} < \frac{d}{30} < \frac{10}{30}$. This means $6 < d < 10$. The integers 7, 8 and 9 satisfy this inequality, so there are three integers.

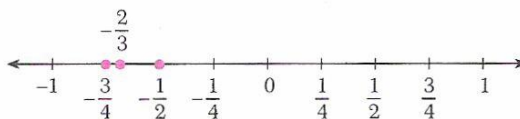
EXAMPLE 73

Order the numbers $a = -\frac{2}{3}$, $b = -\frac{1}{2}$ and $c = -\frac{3}{4}$ and show them on the number line.

Solution The least common multiple of the denominators is 12. Let us equalize the denominators:

$$a = -\frac{2}{3} = -\frac{8}{12}, \quad b = -\frac{1}{2} = -\frac{6}{12}, \quad c = -\frac{3}{4} = -\frac{9}{12}.$$

So $-\frac{9}{12} < -\frac{8}{12} < -\frac{6}{12}$, which means $c < a < b$.

**EXAMPLE 74**

Order each set of rational numbers and give a reason for your order.

a. $\frac{3}{5}, \frac{12}{23}, \frac{4}{9}$ b. $-\frac{30}{23}, -\frac{6}{5}, -\frac{15}{13}$

Solution a. It seems that equalizing the numerators will be easier than equalizing the denominators:

$$\frac{3}{5} = \frac{12}{20}, \quad \frac{12}{23}, \quad \frac{4}{9} = \frac{12}{27}.$$

If two or more fractions have the same numerator then the fraction with the biggest denominator has the smallest value. So $\frac{12}{20} > \frac{12}{23} > \frac{12}{27}$, which is equivalent to $\frac{3}{5} > \frac{12}{23} > \frac{4}{9}$.

b. Again we will equalize the numerators: $-\frac{30}{23}, -\frac{6}{5} = -\frac{30}{25}, -\frac{15}{13} = -\frac{30}{26}$.

Without considering the minus sign, we can write $\frac{30}{23} > \frac{30}{26} > \frac{30}{25}$, which is equivalent to

$$\frac{30}{23} > \frac{15}{13} > \frac{6}{5}. \quad \text{Multiplying each fraction by } -1 \text{ we get } -\frac{30}{23} < -\frac{15}{13} < -\frac{6}{5}.$$



66666
When we multiply all the terms of an inequality by a negative number, we must reverse the inequality sign.

EXAMPLE 75

Arrange the negative integers a, b, c in order if $\frac{2}{ab} = \frac{3}{4bc} = \frac{4}{5ac}$.

Solution If a, b and c are negative then the denominators ab, bc and ac are positive $((-) \cdot (-) = (+))$. Similarly, the product abc is negative.

We can multiply each fraction by abc to get $\frac{2abc}{ab} = \frac{3abc}{4bc} = \frac{4abc}{5ac} = k$, where k must be negative.

Reducing each fraction gives us $2c = \frac{3}{4}a = \frac{4}{5}b = k$. So $c = \frac{k}{2} = \frac{6k}{12}$, $b = \frac{5k}{4} = \frac{15k}{12}$ and $a = \frac{4k}{3} = \frac{16k}{12}$.

Since k is negative, $a < b < c$.



EXAMPLE

76

- a. Find a fraction between $\frac{1}{2}$ and $\frac{3}{4}$. b. Find three fractions between $\frac{1}{3}$ and $\frac{5}{12}$.

Solution a. Equalizing the denominators gives $\frac{1}{2} = \frac{2}{4}$. So finding a number between $\frac{1}{2}$ and $\frac{3}{4}$ is the same as finding a number between $\frac{2}{4}$ and $\frac{3}{4}$: we need to find an integer n such that $\frac{2}{4} < \frac{n}{4} < \frac{3}{4}$. But this does not work since there is no integer between 2 and 3. Therefore we need to build up both fractions so that we can find an integer between the two numerators. Consider building up by 2. Now we need to find a fraction between $\frac{4}{8}$ and $\frac{6}{8}$. Clearly, $\frac{4}{8} < \frac{5}{8} < \frac{6}{8}$. So $\frac{5}{8}$ is an answer.

Of course there are other answers, too. For example, if we build up by 3 we can write $\frac{6}{12} < \frac{7}{12} < \frac{9}{12}$ or $\frac{6}{12} < \frac{8}{12} < \frac{9}{12}$, so $\frac{7}{12}$ and $\frac{8}{12} (= \frac{2}{3})$ are also answers.

- b. Building up both fractions so that each denominator is the least common multiple of denominators will give $\frac{4}{12}$ and $\frac{5}{12}$. But there is no integer between 4 and 5. Therefore we build up both fractions by a convenient integer, for example 4:

$$\frac{1}{3} = \frac{16}{48} < \frac{17}{48} < \frac{18}{48} < \frac{19}{48} < \frac{20}{48} = \frac{5}{12}$$

We can conclude from the previous example that it is always possible to find another rational number between two given rational numbers. This means that there are infinitely many rational numbers between two different rational numbers. This property is called the **density property of rational numbers**. The density property makes it impossible to list the rational numbers in the same way that we list the integers.

The following theorem will help us to find a rational number between two fractions easily.

Theorem

Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}^+$. If $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof

If $\frac{a}{b} < \frac{c}{d}$ then $\frac{ad-bc}{bd} < 0$. Since $bd > 0$, $ad-bc < 0$ or $ad < bc$. (1)

Adding ab to both sides of (1) gives $ab + ad < ab + bc \Leftrightarrow a(b+d) < b(a+c) \Leftrightarrow \frac{a}{b} < \frac{a+c}{b+d}$.

Adding cd to both sides of (1) gives $ad + cd < bc + cd \Leftrightarrow d(a+c) < c(b+d) \Leftrightarrow \frac{a+c}{b+d} < \frac{c}{d}$.



We use the symbol $\Leftrightarrow$ (if and only if) to show that a statement is true in both directions.



EXAMPLE**77**Evaluate $\frac{3ac-bd}{4bd-7ac}$ if $\frac{a}{b} = \frac{4}{3}$ and $\frac{c}{d} = \frac{9}{14}$.**Solution**

If $\frac{a}{b}$ and $\frac{4}{3}$ are equivalent then there is an integer m such that $a = 4m$ and $b = 3m$. Similarly, there is an integer k such that $c = 9k$ and $d = 14k$.

Substituting gives us $\frac{3 \cdot 4m \cdot 9k - 3m \cdot 14k}{4 \cdot 3m \cdot 14k - 7 \cdot 4m \cdot 9k} = \frac{6mk(18-7)}{12mk(14-21)} = \frac{6 \cdot 11}{12 \cdot (-7)} = -\frac{11}{14}$.

EXAMPLE**78**Find the maximum possible value of the quotient $\frac{a+b}{b-a}$ if $3 < a < b < 27$ and $a, b \in \mathbb{Z}$.**Solution**

We must choose the numbers a and b so that the fraction $\frac{a+b}{b-a}$ will have its maximum value.

In other words, $a+b$ must have its maximum value and $b-a$ must have its minimum value.

The minimum possible value of $b-a$ is 1 (note that it must be a positive integer since $a < b$). So we need to choose two numbers a and b such that $a+b$ is maximum, $a-b$ is 1 and $3 < a < b < 27$.

The two numbers are 25 and 26. So the maximum value of $\frac{a+b}{b-a}$ is $\frac{51}{1} = 51$.

Check Yourself 13

1. Order the fractions.

a. $\frac{5}{3}, \frac{23}{12}, \frac{9}{4}$

b. $-\frac{23}{30}, -\frac{13}{15}, -\frac{5}{6}$

2. Find two rational numbers between

a. $-\frac{23}{30}$ and $-\frac{14}{15}$

b. $\frac{3}{7}$ and $\frac{3}{8}$

3. Find a rational number m which satisfies $\frac{a}{3} = \frac{b}{4} = \frac{c}{5} = \frac{2a+3b-mc}{6}$.4. Order the positive integers a, b and c if $\frac{1}{a+b} < \frac{1}{b+c} < \frac{1}{a+c}$.**Answers**

1. a. $\frac{5}{3} < \frac{23}{12} < \frac{9}{4}$

b. $-\frac{13}{15} < -\frac{5}{6} < -\frac{23}{30}$

2. a. $-\frac{24}{30}, -\frac{25}{30}$

b. $\frac{22}{56}, \frac{23}{56}$

3. $\frac{12}{5}$

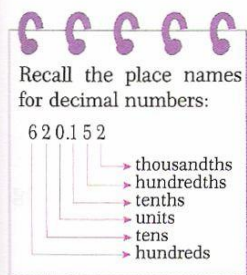
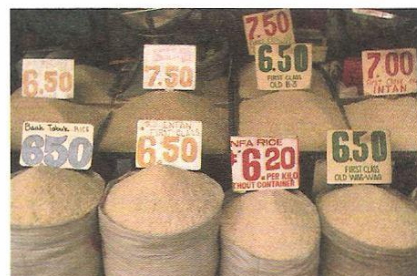
4. $b > a > c$



C. DECIMAL NUMBERS

1. Operations on Decimal Numbers

We learned in our earlier studies that we can express a rational number as a decimal. One way to perform operations involving decimals is to convert all the decimals to fractions and perform the necessary operations with fractions. However, converting decimals to fractions takes time. It is often much easier to perform decimal operations directly. A brief summary of basic operations with decimals is given below.



- To add or subtract two or more decimal numbers, line up the decimal points and add or subtract in the same way as whole numbers. The decimal point is carried straight down to the sum or the difference. Place extra zeros to the right of the decimal points if needed.
- To multiply two decimal numbers, multiply them in the same way as whole numbers. The product has as many decimal places as the total number of the decimal places used in the two numbers.
- To divide a decimal number by another decimal number, first move the decimal point of the divisor so that it becomes a whole number and move the decimal point of the dividend the same number of places to the right. Then divide as if you are working with whole numbers. Place the decimal point of the answer directly below the decimal point in the dividend.

Let us look at some examples.

EXAMPLE

79

Perform the operations.

a. $251.3 + 55.32 + 5.115$

b. $132.5 - 21.528$

c. $25.36 \cdot 12.6$

d. $325.2693 \div 3.21$

Solution

$$\begin{array}{r} \text{a.} \quad 251.\underline{300} \\ \quad 55.\underline{320} \\ + \quad 5.\underline{115} \\ \hline 311.735 \end{array}$$

$$\begin{array}{r} \text{b.} \quad 132.\underline{500} \\ \quad - 21.\underline{528} \\ \hline 110.972 \end{array}$$

$$\begin{array}{r} \text{c.} \quad 25.36 \\ \quad \times 12.6 \\ \hline 15216 \\ \quad 5072 \\ + 2536 \\ \hline 319.536 \end{array}$$

$$\begin{array}{r} \text{d.} \quad 32526.93 \mid 321. \\ \quad - 321 \quad \mid 101.33 \\ \hline \quad 426 \\ \quad - 321 \\ \hline \quad 1059 \\ \quad - 963 \\ \hline \quad 963 \\ \quad - 963 \\ \hline \quad 0 \end{array}$$

EXAMPLE**80**Evaluate $\frac{(\frac{16}{5} - 1.7) \div 0.05}{(\frac{33}{20} - 1.5) \div 1.5}$.**Solution 1**

$$\frac{(\frac{16}{5} - 1.7) \div 0.05}{(\frac{33}{20} - 1.5) \div 1.5} = \frac{(\frac{16}{5} - \frac{17}{10}) \div \frac{5}{100}}{(\frac{33}{20} - \frac{15}{10}) \div \frac{15}{10}} = \frac{\frac{15}{10} \cdot \frac{100}{5}}{\frac{3}{20} \cdot \frac{10}{15}} = \frac{15}{10} \cdot \frac{100}{5} \cdot \frac{20}{3} \cdot \frac{15}{10} = 300$$

Solution 2

$$\frac{(\frac{16}{5} - 1.7) \div 0.05}{(\frac{33}{20} - 1.5) \div 1.5} = \frac{(3.2 - 1.7) \div 0.05}{(1.65 - 1.5) \div 1.5} = \frac{1.5 \div 0.05}{0.15 \div 1.5} = \frac{30}{0.1} = 300$$

2. Repeating Decimals

Definition**repeating decimal**

A decimal number whose decimal part has digits which repeat endlessly is called a **repeating decimal**.

For example, $\frac{5}{11} = 0.454545\dots$ is a repeating decimal. The three dots mean that the digits in the decimal part repeat endlessly.

Notice that when we carry out the division $5 \div 11$, the quotient never ends (or terminates) because zero never appears as a remainder. The block of two digits, 45, keeps appearing and no other digits appear. Often we use a bar ($\overline{}$) to show the block of digits which repeats in a decimal: $0.\overline{45} = 0.454545\dots$

Notation

We write $a.\overline{b}$ to mean a repeating decimal of the form $a.bbb\dots$.

Some books use the notation $a.(b)$ for the repeating decimal $a.bbb\dots$.

EXAMPLE**81**

Convert each fraction to a decimal.

a. $-\frac{16}{3}$

b. $\frac{17}{90}$

c. $3\frac{134}{990}$

Solution

We can convert the fractions by using either a calculator or long division.

a. $-\frac{16}{3} = -5.333\dots = -5.\overline{3}$

b. $\frac{17}{90} = 0.1888\dots = 0.1\overline{8}$

c. $3\frac{134}{990} = 3.1353535\dots = 3.1\overline{35}$

We have already seen how to convert non-repeating decimals such as 6.54 or 9.671 to fractions using multiplication by a power of 10. Converting repeating decimals to fractions is more difficult, since the number of digits in the decimal part is not finite. Let us look at an example.



EXAMPLE

82

Convert each decimal to a fraction.

- a. $2.\overline{5}$ b. $1.\overline{21}$ c. $3.0\overline{2}$

Solution a. If we let $x = 2.\overline{5}$ then $10x = 25.\overline{5}$. So $10x - x = 25.\overline{5} - 2.\overline{5} = 23$, i.e. $9x = 23$.

Remember that we are looking for x . So $x = \frac{23}{9}$.

b. If $x = 1.\overline{21}$ then $10x = 12.\overline{12}$. However, their difference $9x$ does not help since it still contains repeating decimals. So let us consider $100x = 121.\overline{21}$ instead. The difference of $100x$ and x is $99x = 120$ which is free of repeating decimals. So $x = \frac{120}{99} = \frac{40}{33}$.

c. If $x = 3.0\overline{2}$ then $100x = 302.\overline{2}$ and so $99x = 299.2$. So $x = \frac{299.2}{99} = \frac{2992}{990} = \frac{136}{45}$.

Is it true that $1 = 0.9999\dots$?

Let $x = 0.9999\dots$

Then $10x = 9.9999\dots$

$\begin{array}{r} 10x = 9.9999\dots \\ - x = 0.9999\dots \\ \hline \end{array}$

} subtract side by side

$9x = 9$.

Therefore, $x = 1 = 0.9999\dots$

We can see that there is a common strategy for converting repeating decimals to fractions. The steps below show a similar strategy for achieving the same result. We begin with a repeating decimal fraction, for example $61.3\overline{2}$.

1. Suppose that the decimal fraction is an integer (6132).
2. Subtract the non-repeating part from this number ($6132 - 613 = 5519$).
3. Write this difference as the numerator.
4. In the denominator, write as many 9s as there are repeating digits and as many zeros as there are non-repeating digits after the decimal point (one 9 and one zero = 90, so $61.3\overline{2} = \frac{5519}{90}$).

Similarly, $1.2\overline{13} = \frac{1213 - 12}{990} = \frac{1201}{990}$.

CONVERTING A REPEATING DECIMAL TO A FRACTION

decimal number = $\frac{\text{number without the decimal point} - \text{non-repeating part}}{\underbrace{9999\dots9999}_{\text{the number of repeating digits}} \underbrace{0000\dots0000}_{\text{the number of non-repeating digits in the decimal part}}}$



We can write any decimal number which terminates or repeats as a **ratio**. This is why these numbers are called **rational** numbers. But there are an infinite number of decimal numbers which cannot be written as a ratio. For example, we cannot write numbers such as 1.73050807... or 3.1415926535... as a ratio since they do not repeat or terminate. Numbers such as these form the set of **irrational numbers**.

EXAMPLE 83 $a = 0.\overline{1} + 0.\overline{2} + \dots + 0.\overline{9}$ and $b = 0.0\overline{1} + 0.0\overline{2} + \dots + 0.0\overline{9}$ are given. Find $\frac{a-b}{ab}$.

Solution Converting a and b to fractions gives

$$a = \frac{1}{9} + \frac{2}{9} + \dots + \frac{9}{9} = \frac{45}{9} = 5 \text{ and } b = \frac{1}{90} + \frac{2}{90} + \dots + \frac{9}{90} = \frac{45}{90} = \frac{1}{2}.$$

$$\text{So } \frac{a-b}{ab} = \frac{5 - \frac{1}{2}}{5 \cdot \frac{1}{2}} = \frac{\frac{9}{2}}{\frac{5}{2}} = \frac{9}{5}.$$

EXAMPLE 84 a and b are different non-zero digits such that $\frac{0.\overline{ab}}{0.\overline{ba}} = \frac{2}{3}$. Find the ratio $\frac{a}{b}$.

Solution $\frac{0.\overline{ab}}{0.\overline{ba}} = \frac{\frac{ab-a}{90}}{\frac{ba-b}{90}}$, where ab and ba are two-digit numbers.

$$\text{Simplifying gives } \frac{10a+b-a}{10b+a-b} = \frac{9a+b}{a+9b} = \frac{2}{3}.$$

Cross multiplying gives us $27a + 3b = 2a + 18b$, i.e. $25a = 15b$. So $\frac{a}{b} = \frac{3}{5}$.

Check Yourself 14

1. Evaluate the expressions.

a. $\frac{0.002}{0.03} - \frac{0.04}{0.6} + \frac{0.05}{0.002}$ b. $(\frac{0.11}{0.2} + \frac{0.2}{0.11}) \frac{22}{52.1}$

2. Convert each decimal to a fraction.

a. $2.\overline{5}$ b. $0.\overline{112}$ c. $2.12\overline{3}$

3. Evaluate $9.\overline{9} + 8.\overline{8} + 7.\overline{7} + \dots + 2.\overline{2} + 1.\overline{1}$.

Answers

1. a. 25 b. 1 2. a. $\frac{23}{9}$ b. $\frac{112}{999}$ c. $\frac{637}{300}$ 3. 50



EXERCISES 1.3

A. Rational Numbers

1. Convert each fraction to a decimal.

a. $\frac{111}{5}$ b. $\frac{30}{45}$ c. $-\frac{217}{250}$ d. $\frac{117}{78}$

2. Convert each decimal to a fraction.

a. 0.204 b. -3.44 c. 16.17 d. -1.2734

B. Fractions

3. Evaluate the expressions.

a. $\frac{4}{13} + \frac{8}{13}$ b. $-\frac{19}{5} - \frac{12}{5}$
 c. $4\frac{1}{3} - 3\frac{1}{4} + 2\frac{1}{5}$ d. $14 \cdot (\frac{3}{4} - \frac{5}{6}) - 12 - 5$
 e. $\frac{1}{4} - \frac{1}{4} + \frac{3}{4}$ f. $\frac{1}{3} - \frac{1}{3} \cdot \frac{1}{2}$
 g. $42 + 25 \cdot (\frac{1}{4} - \frac{1}{9}) - 34$ h. $\frac{7}{16} \cdot (-\frac{2}{9}) + \frac{13}{8}$

4. Evaluate the expressions.

a. $\frac{1}{12} + (\frac{7}{36} - \frac{5}{24}) \div \frac{1}{6}$
 b. $2 \cdot (\frac{1}{2} + \frac{1}{2} \div 2) \div (1 + \frac{1}{2})^2 - \frac{2}{3}$
 c. $(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}) \div \frac{1}{20}$
 d. $\frac{2}{3} + (\frac{1}{3})^2 \div (\frac{1}{4} - \frac{1}{3} \cdot \frac{333}{666})$

5. Evaluate the expressions.

a. $\frac{\frac{2}{5}}{7} - \frac{2}{\frac{5}{7}}$ b. $\frac{\frac{5}{2}}{\frac{3}{4}}$
 c. $\frac{(3 - \frac{1}{2}) + (1 - \frac{1}{2})}{(4 - \frac{1}{4}) + (\frac{3}{4} - 1)}$ d. $\frac{(-\frac{1}{2})^3 \cdot (-2^5)}{(-2)^2 \cdot 2}$

6. Given $\frac{a}{b} = \frac{c}{d} = 2$, find $\frac{a+b}{b} \cdot \frac{c+d}{c}$.

7. Given $\frac{x}{y} = 3$, find $\frac{2xy}{x^2 + y^2}$.

8. $\frac{a}{c} = \frac{2}{5}$ and $\frac{b}{c} = \frac{5}{13}$ are given. Find $\frac{a+b}{a-b}$.

9. Given $yz = \frac{zx}{2} = \frac{xy}{3}$, find $\frac{x}{yz} + \frac{y}{zx}$.

10. $2x = 3y = 4z$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ are given. Find y .

11. Evaluate

$$(1 - \frac{1}{2^2}) \cdot (1 - \frac{1}{3^2}) \cdot (1 - \frac{1}{4^2}) \cdot \dots \cdot (1 - \frac{1}{19^2}) \cdot (1 - \frac{1}{20^2}).$$

12. $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$, $\frac{1}{a} + \frac{1}{c} = \frac{1}{4}$ and $\frac{1}{b} + \frac{1}{c} = \frac{1}{8}$ are given.

Find $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

13. $m = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{9}$ and $n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ are given.

Find $\frac{m+n}{m-n}$.

14. Evaluate the fractions.

a. $3 + \frac{1}{3 - \frac{4}{4 + \frac{2}{5}}}$

b. $\frac{5}{1 - \frac{4}{2 - \frac{3}{4 + \frac{2}{7}}}}$

c. $1 - \frac{\frac{1}{5}}{1 - \frac{1}{1 - \frac{1}{1 - \frac{1}{5}}}}$

d. $\frac{\frac{7}{3} - \frac{4}{6}}{1 - \frac{11}{12}} \div \frac{\frac{7}{4} + \frac{2}{3} - 1}{\frac{7}{6} \cdot \frac{5}{3}} - \frac{\frac{2}{9} + \frac{1}{3}}{\frac{3}{7} \cdot \frac{7}{3} + \frac{2}{3}}$

15. Find the value of m which satisfies

$$\frac{1}{2 + \frac{1}{2 + \frac{m}{1 + \frac{1}{5 + \frac{1}{1 + \frac{1}{3}}}}}} = \frac{1}{4}.$$

16. $m = -\frac{2}{3}$, $n = -\frac{1}{5}$ are given.

Evaluate $\frac{m}{\frac{-1}{m} + 1} - \frac{n}{1 - \frac{n}{m}} - \frac{2}{\frac{1}{m} - \frac{1}{n}}$.

17. Evaluate the fractions.

a. $5 + \frac{6}{5 + \frac{6}{5 + \frac{6}{\ddots}}}$

b. $3 - \frac{2}{3 - \frac{2}{3 - \frac{2}{\ddots}}}$

c. $3 + \frac{4}{3 + \frac{4}{3 + \frac{4}{\ddots}}}$

d. $2 + \frac{8}{2 + \frac{8}{2 + \frac{8}{\ddots}}}$

18. Find the value of x which satisfies

$$\frac{3x}{2 + \frac{3}{2 + \frac{3}{\ddots}}} - \frac{4}{3 - \frac{2}{3 - \frac{2}{\ddots}}} = 4.$$

19. A fraction is equivalent to $\frac{2}{3}$ when 8 is subtracted from both its numerator and its denominator. It is equivalent to $\frac{20}{29}$ when 6 is subtracted from its numerator and its denominator. Find the fraction.

20. Given that $\frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \dots + \frac{19}{20} = m$,

find $\frac{23}{6} + \frac{27}{7} + \frac{31}{8} + \dots + \frac{79}{20}$ in terms of m .



21. Find the smallest fraction in this list:

$$\frac{17}{20}, \frac{7}{11}, \frac{5}{7}, \frac{7}{8}, \frac{13}{15}$$

22. Order the numbers $a = -\frac{12}{17}$, $b = -\frac{102}{107}$ and $c = -\frac{1002}{1007}$.

23. Find three numbers between $\frac{8}{5}$ and $\frac{11}{3}$.

24. Order each set of rational numbers, from the smallest to the largest.

a. $0, -\frac{2}{5}, -\frac{5}{8}$

b. $\frac{4}{9}, \frac{3}{8}, \frac{5}{6}$

c. $\frac{6}{11}, 1\frac{2}{3}, \frac{8}{13}$

d. $-1\frac{1}{3}, -2\frac{1}{4}, -\frac{7}{6}$

e. $-\frac{8}{3}, -\frac{7}{5}, -\frac{6}{7}, -1\frac{1}{4}$

25. a is a positive integer. Order the numbers

$$\frac{a-1}{a}, \frac{a}{a+1} \text{ and } \frac{a+1}{a+2}$$

26. Order the rational numbers

$$A = \frac{314871253}{932106715} \text{ and } B = \frac{314871254}{932106717}$$

C. Decimal Numbers

27. Evaluate the expressions.

a. $1.2 + 0.02$

b. $0.02 - 0.7$

c. $112.3 - 5.4 + 2.14$

d. $-3.114 + 7.12 - 1.9$

e. $5 + 4.774 + 0.00001$

28. Evaluate the expressions.

a. $1.2 \cdot 0.5$

b. $2.5 \cdot (-3.12)$

c. $25.6878 \div 6.7$

d. $-6.2 \div 3.14 \cdot 12.12$

29. Evaluate the expressions.

a. $\frac{5}{0.5} - \frac{4}{0.25} + \frac{1}{0.125}$

b. $\frac{0.000077 + 0.0077 + 0.77}{0.000111}$

c. $\frac{0.2}{0.02} + \frac{0.1^2}{0.01} + \frac{5}{0.5} - \frac{0.4^2}{0.04}$

30. yx is a two-digit number and $0.y$ is a decimal number such that $\frac{x}{0.y} + m = \frac{yx}{0.y}$. Find m .

31. Convert each repeating decimal to a fraction.

a. $-3.\overline{235}$

b. $0.04\overline{556}$

c. $78.\overline{1024}$

32. Evaluate $\frac{3.\overline{3}}{1.\overline{1} + 2.\overline{2}}$.

33. Evaluate $\frac{0.\overline{3}}{1.\overline{9} + \frac{1}{2.\overline{9} + \frac{3}{0.6}}}$.

34. Simplify $\frac{\frac{1}{0.\overline{xx}} + \frac{1}{0.\overline{yy}}}{\frac{1}{y} + \frac{1}{x}}$, where x and y are digits.

35. x and y are two real numbers such that
 $x = 1.\overline{1} + 1.\overline{2} + \dots + 1.\overline{9}$ and
 $y = 1.0\overline{1} + 1.0\overline{2} + \dots + 1.0\overline{9}$.
 Evaluate $\frac{xy}{x-y}$.

36. Simplify $\frac{0.\overline{xy} + 0.\overline{ab}}{0.\overline{xy} + 0.\overline{ab}}$, where x, y, a and b are digits.

37. Evaluate $\frac{0.\overline{3} + 0.\overline{44} + 0.\overline{555} + 0.\overline{6666}}{0.\overline{7} + 0.\overline{88} + 0.\overline{999} + 0.\overline{2222}}$.

38. Simplify $\frac{x.\overline{x} + xx.\overline{xx} + xxx.\overline{xxx}}{xxx.x + xx.xx + x.xxx}$, where x is a digit.

Mixed Problems

39. $\frac{xy}{x+y} = a$, $\frac{xz}{x+z} = b$ and $\frac{yz}{y+z} = c$.

Find x in terms of a, b and c .

40. If x men can finish a job in d days, how many days would it take $x + y$ men to finish the same job?

41. Evaluate $\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{100 \cdot 101}$.

42. Evaluate $3 - \frac{2}{3 - \frac{2}{3 - \frac{2}{\ddots}}}$ if the division repeats one hundred times.

43. Evaluate $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots + \frac{19}{9^2 \cdot 10^2}$.

44. Order $A = \frac{11877}{35632}$ and $B = \frac{23752}{71258}$.

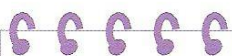
45. Let $x = 0.02468101214 \dots 100102104$. The digits of x after the decimal point are obtained by writing the even integers between zero and 104. Find the 101st digit to the right of the decimal point.



4

REAL NUMBERS

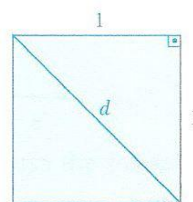
A. IRRATIONAL NUMBERS



The Pythagorean Theorem states that the square of the length of the hypotenuse h of a right triangle is the sum of the squares of the other two sides a and b :

$$a^2 + b^2 = h^2.$$

Consider the following problem in elementary geometry: Find the length of the diagonal of a square with a side length of 1 unit.



If we call the length of the diagonal d , then by the Pythagorean Theorem, $d^2 = 1^2 + 1^2 = 2$. So d is the square root of 2: $d = \sqrt{2}$. But what is the exact value of d ? We can say it is approximately 1.41. In fact it is a little greater than 1.412 and a little less than 1.422. A better approximation is $\sqrt{2} = 1.41421356\dots$. This decimal number does not terminate or repeat. Therefore, it is impossible to write it as a ratio of two integers, and so we cannot find the exact value of the number. We call numbers such as this **irrational** numbers.

Definition

irrational number

A number that cannot be expressed as a ratio of two integers is called an **irrational number**. The set of irrational numbers is denoted by $\mathbb{Q}'$ or I .

For example, square roots such as $\sqrt{12}$, $\sqrt{3}$ and $\sqrt{7}$, and the number $\pi = 3.141592653\dots$ are irrational numbers. However, $\sqrt{16}$ is not an irrational number, since $\sqrt{16} = 4$.

EXAMPLE

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Prove that $\sqrt{2}$ is not a rational number.

Solution

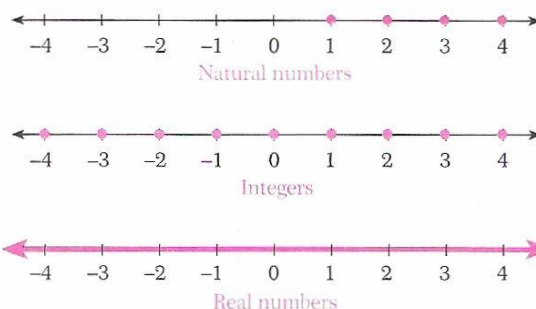
Let us assume that $\sqrt{2}$ is a rational number and try to find a contradiction. So we can write $\sqrt{2} = \frac{a}{b}$ such that a and b are **relatively prime** positive integers (i.e. $\frac{a}{b}$ is a fraction reduced to its lowest terms).

Taking the square of both sides gives us $2 = \frac{a^2}{b^2}$, i.e. $a^2 = 2b^2$. This means that $a^2 = a \cdot a$ is an even number. If $a \cdot a$ is even, then a is even. This means that $a = 2k$ for some $k \in \mathbb{N}$. So we can rewrite $a^2 = 2b^2$ as $4k^2 = 2b^2$. This simplifies to $2k^2 = b^2$, which means that b is also even, i.e. $b = 2m$ ($m \in \mathbb{N}$). But if $a = 2k$ and $b = 2m$ then 2 is a common divisor of a and b . So a and b are **not relatively prime**, which contradicts our initial assumption. Therefore we cannot write the number $\sqrt{2}$ in the form $\frac{a}{b}$. In other words, $\sqrt{2}$ is not a rational number.

B. REAL NUMBERS

1. Understanding Real Numbers

We have seen that we can show every natural number, integer and fraction as a point on the number line. In other words, every rational number corresponds to a point on the number line. But there are still points on the number line that do not correspond to rational numbers. These points correspond to irrational numbers. Rational numbers together with irrational numbers 'fill up' the number line. The set of



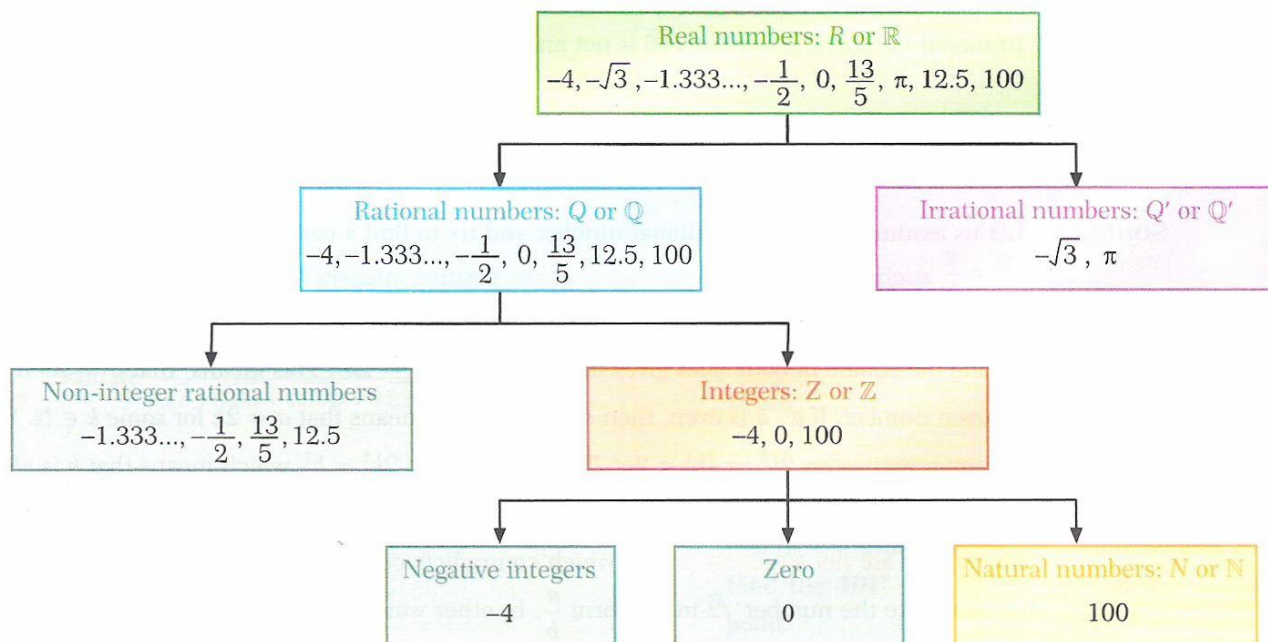
numbers that corresponds to all the points on the number line is called the set of **real numbers**.

Definition

real number

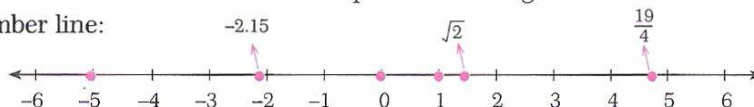
The set of rational numbers and the set of irrational numbers together form the **set of real numbers**, denoted by $\mathbb{R}$. In other words, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

Any number that we have encountered so far in this book and in our studies of other topics in algebra is a real number. Note that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. Sometimes the letters N , Z , Q and R are also used to denote the number sets. The diagram below shows how the number sets are related.



2. The Real Number Line

There is a one-to-one correspondence between the set of real numbers and the set of points on a line. Imagine a line which is infinitely long, marked with a point 0 (zero). Each real number corresponds to exactly one point on this line, and each point on this line corresponds to exactly one real number. This line is called the **real number line**. On the real number line, the point with coordinate zero is called the **origin**. All the numbers to the right of zero are called **positive real numbers**, and the numbers to the left of zero are called **negative real numbers**. The real number zero is neither positive nor negative. Look at some numbers on the real number line:



To show the relationship between two numbers which are not equal we use the following inequality symbols.

Symbol	Meaning	Example
$<$	is less than	$-2.25 < 7$
$>$	is greater than	$\sqrt{7} > 2$
$\leq$	is less than or equal to	$0 \leq 0.12$
$\geq$	is greater than or equal to	$1.333... \geq -2$
$\neq$	is not equal to	$3 \neq 4$
$\approx$	is approximately equal to	$\sqrt{2} \approx 1.41$

We can rewrite an inequality with the inequality symbol reversed: $3 < 5$ is the same as $5 > 3$. As we saw in our study of integers, if a and b are two numbers on the number line then the number which lies furthest to the right is greater. In other words,

- if a lies to the left of b on the number line then $a < b$.
- if a lies to the right of b on the number line then $a > b$.

There are three ways to express an inequality mathematically: using **inequality notation**, using **interval notation** and by drawing a **graph**. For example, we can write the set of all real numbers x that are less than 3 as follows:

$$\begin{array}{ll}
 x < 3 & \text{(inequality notation)} \\
 (-\infty, 3) & \text{(interval notation)} \\
 \text{---} \leftarrow \bullet \text{---} & \text{(graph)} \\
 & 3
 \end{array}$$

These are three different ways of writing the same set of numbers.

Note

The symbol ∞ (infinity) is not a real number. We use it to show that an interval extends infinitely far to the right (∞) or to the left ($-\infty$) on the real number line.

The set of all real numbers which are less than 3 extends up to (but does not include) 3 on the number line. It extends to infinity in the opposite direction, because there are an infinite number of real numbers less than 3. We say that 3 and ∞ (infinity) are the **boundaries** of this set. An interval which has infinity as a boundary is called an **unbounded interval**. The chart below shows different types of unbounded interval on the real number line.

Unbounded intervals			
Interval type	Inequality	Interval	Graph
closed	$x \geq a$	$[a, \infty)$	
open	$x > a$	(a, ∞)	
closed	$x \leq a$	$(-\infty, a]$	
open	$x < a$	$(-\infty, a)$	
open	$-\infty < x < \infty$	$(-\infty, \infty)$	

An interval which does not have infinity as a boundary is called a **bounded interval**. The chart below shows some bounded intervals on the real number line.

Bounded intervals			
Interval type	Inequality	Interval	Graph
closed	$a \leq x \leq b$	$[a, b]$	
open	$a < x < b$	(a, b)	
half-open	$a \leq x < b$	$[a, b)$	
half-open	$a < x \leq b$	$(a, b]$	

EXAMPLE 86 Graph each inequality and write it using interval notation.

- a. $x \leq 1$ b. $-1 \leq x < 2$ c. $x > -4$

Solution a. $x \leq 1$ means all real numbers less than or equal to 1.

The graph is . The interval is $(-\infty, 1]$.

- b. $-1 \leq x < 2$ means all real numbers less than 2 and greater than or equal to -1.

$[-1, 2)$

- c. $x > -4$ means all real numbers greater than -4.

$(-4, \infty)$

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a.  b.  c. 

a. This half-open interval corresponds to the inequality $-1 < x \leq 4$ and the interval $(-1, 4]$.

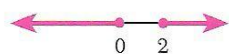
b. This open interval corresponds to the inequality $2 < x$ and the interval $(2, \infty)$.

c. This closed interval corresponds to the inequality $-1 \leq x \leq 2$ and the interval $[-1, 2]$.

A combination of two or more intervals is called a **union**. For example, we can show the set of all real numbers x that are less than zero **or** greater than 2 as follows:

$x < 0$ or $x > 2$ (inequality notation)

$(-\infty, 0) \cup (2, \infty)$ (interval notation as a union)



(graph)

Sometimes two intervals overlap. The set of numbers which lie in two or more intervals is called the **intersection** of the intervals. For example, we can show the set of all real numbers x that are less than 3 and greater than or equal to -1 **and** less than zero as follows:

$-1 \leq x < 3$ and $x < 0$ (inequality notation)

$$[-1, 3) \cap (-\infty, 0) = [-1, 0) \quad (\text{interval notation as an intersection})$$



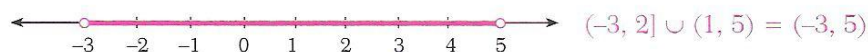
(graph)

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Graph each interval on the number line and write it as an inequality.

a. $(-3, 2] \cup (1, 5)$ b. $(-3, 5) \cap (1, 5]$

a. We graph each interval separately and then find their union.



We can write this union as $-3 < x < 5$.

b. We graph each interval separately and then find their intersection.

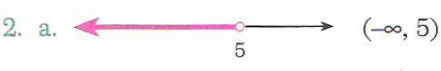


We can write this intersection as $1 < x < 5$.

Check Yourself 15

- Graph each interval on a number line and express it as an inequality.
 - $(-3, \infty)$
 - $(1, 2]$
 - $[-1, 1) \cup (0, 4)$
 - $[-1, 1) \cap (0, 4)$
- Graph each inequality on a number line and express it as an interval.
 - $x \leq 5$
 - $2 \leq x < 4$
 - $-3 < x \leq 3$
 - $x \geq 4$

Answers

-  $x > -3$
 -  $1 < x \leq 2$
 -  $-1 \leq x < 4$
 -  $0 < x < 1$
-  $(-\infty, 5)$
 -  $[2, 4)$
 -  $(-3, 3]$
 -  $[4, \infty)$

3. Properties of Real Numbers

We will now look at some of the basic properties of real numbers. We are already familiar with most of them from our previous studies. However, since real numbers cover all the number sets we have seen so far, a final review of their basic properties will be useful.

For $x, y, z \in \mathbb{R}$ the following properties hold.

Properties of addition in the set of real numbers		
Property	Explanation	Example
closure	$x + y$ is a unique element of $\mathbb{R}$.	$\frac{4}{5} \in \mathbb{R}, \sqrt{2} \in \mathbb{R} \Rightarrow \frac{4}{5} + \sqrt{2} \in \mathbb{R}$
commutative	$x + y = y + x$	$5.21 + (-2) = (-2) + 5.21$
associative	$(x + y) + z = x + (y + z)$	$(2 + \frac{5}{9}) + \sqrt{2} = 2 + (\frac{5}{9} + \sqrt{2})$
identity	zero is the additive identity: $0 + x = x + 0 = x$.	$1.\bar{3} + 0 = 0 + 1.\bar{3} = 1.\bar{3}$
inverse	$-x$ is the additive inverse of x : $x + (-x) = (-x) + x = 0$.	$\frac{11}{3} + (-\frac{11}{3}) = (-\frac{11}{3}) + \frac{11}{3} = 0$



Properties of multiplication in the set of real numbers		
Property	Explanation	Example
closure	xy is a unique element of $\mathbb{R}$.	$-4 \in \mathbb{R}, \frac{34}{23} \in \mathbb{R} \Rightarrow (-4) \cdot \frac{34}{23} \in \mathbb{R}$
commutative	$xy = yx$	$5 \cdot \sqrt{3} = \sqrt{3} \cdot 5$
associative	$(xy)z = x(yz)$	$(\sqrt{7} \cdot 3.6\bar{5}) \cdot 40 = \sqrt{7}(3.6\bar{5} \cdot 40)$
identity	1 is the multiplicative identity: $x \cdot 1 = 1 \cdot x = x$.	$\frac{37}{2} \cdot 1 = 1 \cdot \frac{37}{2} = \frac{37}{2}$
inverse	$\frac{1}{x}$ is the multiplicative inverse of x : $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$.	$\sqrt{5} \cdot \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \sqrt{5} = 1$
distributive	$x(y + z) = xy + xz$	$\frac{1}{2}(2.25 + \pi) = \frac{1}{2} \cdot 2.25 + \frac{1}{2} \cdot \pi$

4. Numerical Inequalities

Up to now we have defined inequality using the number line: $2 < 3$ since 2 is to the left of 3 on the number line. More generally, we say that a number a is less than a number b if a is located to the left of b on the number line. However, this is the geometrical approach to defining inequality. We need an algebraic definition in order to deal with further problems. Algebraically we can say that $2 < 3$ since $2 - 3$ is negative. More generally, if $a, b \in \mathbb{R}$ then the following statements are true:

- $a > b$ if and only if $a - b > 0$
- $a < b$ if and only if $a - b < 0$.



EXAMPLE 89 Given that $a < 0$ and $b > 0$, determine whether each expression is positive or negative.

- a. $3a - b$ b. $4a(b + 4)$

Solution a. $b > 0$ means $-b < 0$. If $3a < 0$ and $-b < 0$ then $3a + (-b) < 0$. So $3a - b < 0$.
b. If $b + 4 > 0$ and $4a < 0$ then $4a(b + 4) < 0$.

EXAMPLE 90

Given $a \in \mathbb{R}$, compare $(a - 6)^2 + 1$ with $(a - 5)(a - 7)$.

Solution To compare the two expressions we will consider their difference. If the difference is positive then the first expression will be greater, otherwise the first expression will be smaller.

$$((a - 6)^2 + 1) - ((a - 5)(a - 7)) = (a^2 - 12a + 37) - (a^2 - 12a + 35) = 37 - 35 > 0$$

So the first expression is bigger and therefore $(a - 6)^2 + 1 > (a - 5)(a - 7)$.

EXAMPLE 91

$0 < a < b$ is given. Prove that $\frac{1}{b} < \frac{1}{a}$.

Solution Consider the difference $\frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}$. Since $a - b < 0$ and $ab > 0$, we have $\frac{a-b}{ab} < 0$.

So $\frac{1}{b} - \frac{1}{a} < 0$, which means $\frac{1}{b} < \frac{1}{a}$.

PROPERTIES OF ORDER

Let $a, b, c, d \in \mathbb{R}$. Then the following statements are true.

1. If $a > b$ and $b > c$ then $a > c$.
2. If $a > b$ then $(a + c) > (b + c)$ and $(a - c) > (b - c)$.
3. a. If $c > 0$ and $a > b$ then $ac > bc$.
b. If $c < 0$ and $a > b$ then $ac < bc$.
4. If $a > b$ and $c > d$ then $(a + c) > (b + d)$.

Proof 1. If $a > b$ and $b > c$ then $(a - b) > 0$ and $(b - c) > 0$. We know that the sum of two positive numbers is positive. So $(a - b) + (b - c) > 0$, which means $(a - c) > 0$. So $a > c$.

2. If $a > b$ then $(a - b) > 0$, and so $(a - b) + (c - c) > 0$. (Note that since $c - c = 0$, adding it to the left-hand side of the inequality does not change anything.) Rearranging this inequality gives us $(a + c) - (b + c) > 0$, i.e. $(a + c) > (b + c)$.

We can prove $(a - c) > (b - c)$ in the same way.

3. a. If $a > b$ then $(a - b) > 0$. Since we also know that $c > 0$ and the product of two positive numbers is positive, we have $(a - b)c > 0$, i.e. $(ac - bc) > 0$. This means $ac > bc$.

b. If $a > b$ then $(a - b) > 0$. Since we also know that $c < 0$ and the product of a positive and a negative number is negative, we have $(a - b)c < 0$, i.e. $(ac - bc) < 0$. This means $ac < bc$.

4. If $a > b$ and $c > d$ then $(a - b) > 0$ and $(c - d) > 0$. The sum of two positive numbers is positive. So $(a - b) + (c - d) > 0$, i.e. $(a + c) - (b + d) > 0$. This means $(a + c) > (b + d)$.



EXAMPLE**92**Given $3 \leq x < 9$, find all possible values of each expression.

a. $x + 5$

b. $-0.5x$

c. $2x - 3$

Solution

- a. By the second property of order, we can add the same number to all parts of the inequality without changing the inequality:

$$3 \leq x < 9$$

$$3 + 5 \leq x + 5 < 9 + 5$$

$$8 \leq x + 5 < 14.$$

- b. By the third property of order, we can multiply all parts of the inequality by the same number (when the number is negative, the inequality signs are reversed):

$$3 \leq x < 9$$

$$-0.5 \cdot 3 \geq -0.5 \cdot x > -0.5 \cdot 9$$

$$-1.5 \geq -0.5x > -4.5.$$

- c. We can use the third property to multiply all parts of the inequality by 2, and then the second property to subtract 3 from all parts of the result:

$$3 \leq x < 9$$

$$2 \cdot 3 \leq 2 \cdot x < 2 \cdot 9$$

$$6 \leq 2x < 18$$

$$6 - 3 \leq 2x - 3 < 18 - 3$$

$$3 \leq 2x - 3 < 15.$$

EXAMPLE**93**
 $-3 < x < 4$ and $2 < y < 5$ are given. Find all possible values of $4x - 3y$.
Solution

We first multiply the first inequality by 4 (to get $4x$) and the second inequality by -3 (to get $-3y$). Since -3 is negative, we must remember to reverse the inequality signs in the second inequality.

$$-3 < x < 4$$

$$2 < y < 5$$

$$4 \cdot (-3) < 4 \cdot x < 4 \cdot 4$$

$$-3 \cdot 2 > -3 \cdot y > -3 \cdot 5$$

$$-12 < 4x < 16$$

$$-6 > -3y > -15$$

We can rewrite the second result as $-15 < -3y < -6$. By the fourth property of order, we can add this to the first result without changing the inequality:

$$-12 < 4x < 16$$

$$+ \quad -15 < -3y < -6$$

$$-27 < (4x - 3y) < 10.$$

EXAMPLE 94

Prove that $\frac{x^2 + y^2}{2} \geq xy$ for any real numbers x and y .

Solution Consider the difference $\frac{x^2 + y^2}{2} - xy = \frac{x^2 + y^2 - 2xy}{2} = \frac{(x - y)^2}{2}$.

Now $(x - y)^2 \geq 0$ (since it is a square), and so $\frac{(x - y)^2}{2} \geq 0$.

So $\frac{x^2 + y^2}{2} - xy \geq 0$, which means $\frac{x^2 + y^2}{2} \geq xy$.

EXAMPLE 95

Prove that $(x^2 + y^2 + z^2) > (xy + yz + zx - 2)$ for any real numbers x , y and z .

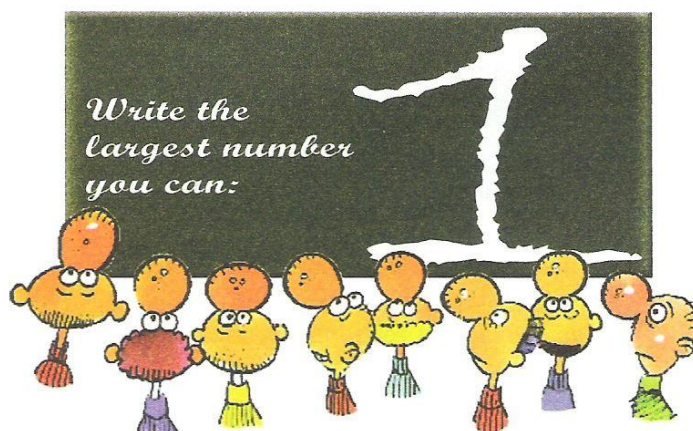
Solution From the previous example we know that $\frac{x^2 + y^2}{2} \geq xy$, $\frac{y^2 + z^2}{2} \geq yz$ and $\frac{z^2 + x^2}{2} \geq zx$.

By the fourth property of order we can add these inequalities side by side to get

$$\frac{x^2 + y^2}{2} + \frac{y^2 + z^2}{2} + \frac{z^2 + x^2}{2} \geq xy + yz + zx, \text{ i.e. } (x^2 + y^2 + z^2) \geq (xy + yz + zx).$$

Clearly, $(xy + yz + zx) > (xy + yz + zx - 2)$.

So by the first property of order we have $(x^2 + y^2 + z^2) > (xy + yz + zx - 2)$.

**Check Yourself 16**

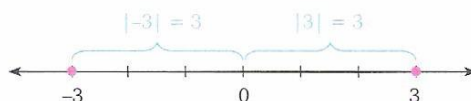
- Determine whether $\frac{3-a}{2b}$ is positive or negative if $a < 0$ and $b > 0$.
- Prove that $[(2a + 5)(2a - 5)] - (3a - 2)^2 \leq [3(4a - 9)] - 2$.
- $-1 < x < 4$ is given. Find all possible values of $-3x + 1$.

Answers

- positive
- Hint: Take the difference of the two sides.
- $(-11, 4)$

C. ABSOLUTE VALUE

The **absolute value** of any real number x , denoted by $|x|$, is the distance between x and zero on the number line. For example, -3 and 3 have the same absolute value since they are the same distance from zero on the number line. We can write $|-3| = |3| = 3$.



Definition

absolute value of a number

For any real number x , the **absolute value** of x is denoted by $|x|$ and $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.

We can understand the above definition as follows: If x is positive or zero then the absolute value of x is x . If x is negative then the absolute value of x is $-x$ (which is a positive number). For example, $|12| = 12$, $|-7| = -(-7) = 7$ and $|0| = 0$.

Note

1. $|-x| = |x|$
2. The absolute value of a number is never negative.

EXAMPLE

96

Given $x = -3$ and $y = 2$, evaluate $|-2x + y| - |x - 2y| - |-xy|$.

Solution

$$\begin{aligned} |-2x + y| - |x - 2y| - |-xy| &= |-2(-3) + 2| - |(-3) - 2 \cdot 2| - | -(-3) \cdot 2| \\ &= |6 + 2| - |-3 - 4| - |3 \cdot 2| \\ &= |8| - |-7| - |6| \\ &= 8 - 7 - 6 = -5 \end{aligned}$$

EXAMPLE

97

Find all the values of x which satisfy each equation.

- a. $|x| = 2$ b. $|x + 2| = 5$ c. $|2x - 3| = 0.5$

- Solution**
- If $|x| = 2$ then either $x = 2$ or $x = -2$.
 - If $|x + 2| = 5$ then either $x + 2 = 5$ or $x + 2 = -5$. So $x = 3$ or $x = -7$.
 - If $|2x - 3| = 0.5$ then either $2x - 3 = 0.5$ or $2x - 3 = -0.5$. So $x = 1.75$ or $x = 1.25$.

EXAMPLE 98 Given $a < 0$, simplify $a + |a| + |-a|$.

Solution Let us consider each term of the sum independently.

The first term a does not use the absolute value so it needs no further operation: $a = a$.

The second term $|a|$ has a inside the absolute value sign. Since a is negative, its absolute value is $-a$: $|a| = -a$.

The second term $|-a|$ has $-a$ inside the absolute value sign. Since $-a$ is positive, its absolute value does not change: $|-a| = -a$.

Combining these results gives $a + |a| + |-a| = a + (-a) + (-a) = -a$.

EXAMPLE 99 Given $x > 2$, simplify $|1 - x| + |x + 2|$.

Solution If $x > 2$ then $1 - x < 0$ and $x + 2 > 0$.

So $|1 - x| + |x + 2| = x - 1 + x + 2 = 2x + 1$.

EXAMPLE 100 Given $x < 0 < y$, simplify $|x - y| + |x| - |y|$.

Solution If $x < y$ then $x - y < 0$. So $|x - y| = -(x - y)$.

If $x < 0$ and $y > 0$ then $|x| = -x$ and $|y| = y$.

So $|x - y| + |x| - |y| = -(x - y) + (-x) - y = -2x$.

EXAMPLE 101 Given $x < y < 0$, simplify $|x + |x + y|| + |x| - |y|$.

Solution If $x < 0$ and $y < 0$ then $x + y < 0$. So $|x + y| = -(x + y)$.

If $x < 0$ and $y < 0$ then $|x| = -x$ and $|y| = -y$.

So $|x + |x + y|| + |x| - |y| = |x - (x + y)| - x - (-y) = |-y| - x + y$
 $= -y - x + y$ (since $-y > 0$)
 $= -x$.

EXAMPLE 102 For what value of $\frac{a+b}{a-b}$ does $|3a - 2b|$ have its minimum value?

Solution Since an absolute value expression can never be negative, the minimum value of $|3a - 2b|$ is 0.

So $|3a - 2b| = 0$ which gives $3a - 2b = 0$, i.e. $a = \frac{2b}{3}$.

Substituting gives $\frac{a+b}{a-b} = \frac{\frac{2b}{3} + b}{\frac{2b}{3} - b} = \frac{\frac{5b}{3}}{-\frac{b}{3}} = \frac{5b}{3} \cdot \frac{-3}{b} = -5$.



PROPERTIES OF THE ABSOLUTE VALUE

Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then the following properties hold.

1. $|a - b| = |b - a|$ (e.g. $|5 - 2| = |2 - 5| = 3$)
2. $|ab| = |a| \cdot |b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (e.g. $|3 \cdot (-2)| = |3| \cdot |-2| = 6$ and $\left|\frac{-20}{-4}\right| = \frac{|-20|}{|-4|} = 5$)
3. $|a^n| = |a|^n$ (e.g. $|(-2)^5| = |(-2)|^5 = 32$)
4. $|a + b| \leq |a| + |b|$
(e.g. $|5 + (-2)| = 3$ and $|5| + |-2| = 7$, therefore $|5 + (-2)| \leq |5| + |-2|$). This property is called the triangle inequality.

- Proof**
1. • If $a - b \geq 0$ then $|a - b| = a - b$ and $|b - a| = -(b - a)$. So $|a - b| = |b - a|$.
• If $a - b < 0$ then $|a - b| = -(a - b)$ and $|b - a| = b - a$. So $|a - b| = |b - a|$.
So for any value of a and b the equality is always true.
 2. Let us prove that $|ab| = |a| \cdot |b|$ by considering each possible case.
 - If $a \geq 0$ and $b \geq 0$ then $|ab| = ab$ and $|a| \cdot |b| = ab$. So $|ab| = |a| \cdot |b|$.
 - If $a \geq 0$ and $b < 0$ then $|ab| = -ab$ and $|a| \cdot |b| = a \cdot (-b)$. So $|ab| = |a| \cdot |b|$.
 - If $a < 0$ and $b \geq 0$ then $|ab| = -ab$ and $|a| \cdot |b| = (-a) \cdot b$. So $|ab| = |a| \cdot |b|$.
 - If $a < 0$ and $b < 0$ then $|ab| = ab$ and $|a| \cdot |b| = (-a) \cdot (-b)$. So $|ab| = |a| \cdot |b|$.
 Since there is no other possible case, $|ab| = |a| \cdot |b|$ is always true.
We can prove $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ in a similar way.
 3. When $n = 2$ we have $|a^2| = |a|^2$.
This is true since by property 2 above, $|a^2| = |a \cdot a| = |a| \cdot |a| = |a|^2$.
This can be generalized for any n since $|a^n| = \underbrace{|a \cdot a \cdot a \cdots a|}_{n \text{ times}} = \underbrace{|a| \cdot |a| \cdot |a| \cdots |a|}_{n \text{ times}} = |a|^n$.
 4. • If $a \geq 0$ and $b \geq 0$ then $|a + b| = a + b$ and $|a| + |b| = a + b$.
• If $a < 0$ and $b < 0$ then $|a + b| = -(a + b)$ and $|a| + |b| = (-a) + (-b)$.
So for the above cases, $|a + b| = |a| + |b|$.
• If $a \geq 0$ and $b < 0$ then $|a| = a$ and $|b| = -b$.
Now, either $|a + b| = -(a + b) = (-a) + (-b) = -|a| + |b| < |a| + |b|$,
or $|a + b| = a + b = a - (-b) = |a| - |b| < |a| + |b|$.
We can prove the case when $a < 0$ and $b \geq 0$ in a similar way.
Since there is no other possible case, $|a + b| \leq |a| + |b|$ is always true.

EXAMPLE 103 Simplify $\frac{|-x^{2006}|}{|(-x)^{2006}|} + \frac{|yz-yx|}{|y||x-z|}$.

Solution We can use the first, second and third properties of the absolute value together in this question.

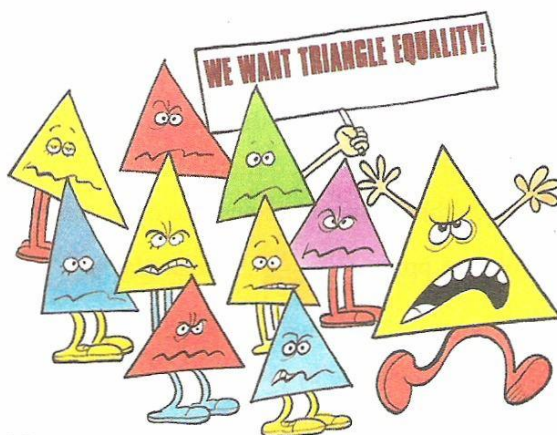
$$\frac{|-x^{2006}|}{|(-x)^{2006}|} + \frac{|yz-yx|}{|y||x-z|} = \frac{|-x^{2006}|}{|-x^{2006}|} + \frac{|y(z-x)|}{|y||x-z|} = 1 + \frac{|y| \cdot |z-x|}{|y| \cdot |x-z|} = 1 + \frac{|y| \cdot |z-x|}{|y| \cdot |z-x|} = 1 + 1 = 2$$

EXAMPLE 104 Find the minimum value of $\frac{1}{2} + \frac{|x|+|y|}{|x+y|}$.

Solution By the triangle inequality, $|x+y| \leq |x| + |y|$.

Since $|x+y|$ and $|x| + |y|$ are positive, the value of the fraction $\frac{|x|+|y|}{|x+y|}$ increases as $|x+y|$ decreases. So at the minimum value of $\frac{|x|+|y|}{|x+y|}$, $|x+y|$ has its maximum value: $|x+y| = |x| + |y|$.

So the minimum value of $\frac{1}{2} + \frac{|x|+|y|}{|x+y|}$ is $\frac{1}{2} + 1 = \frac{3}{2}$.



Check Yourself 17

- Given $x = 1$ and $y = -4$, find $||x+y| - |3-x+y|| + |2x+3y|$.
- Given $0 < (b+c) < a$, simplify $|b+c-a| + |b+c|$.
- Given $x < y < 0$, simplify $|x + |x+y|| + |x| - |y| - |x-y|$.
- Given $a < b < 0 < c$, simplify $|ac-bc| - |ac-ab| + |ab-cb|$.
- Simplify $\frac{|(x-y)^{100}|}{|y-x|^{100}}$.

Answers

1. 11 2. a 3. $-y$ 4. 0 5. 1

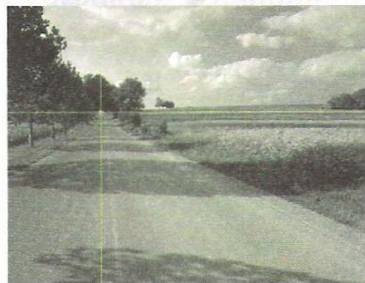
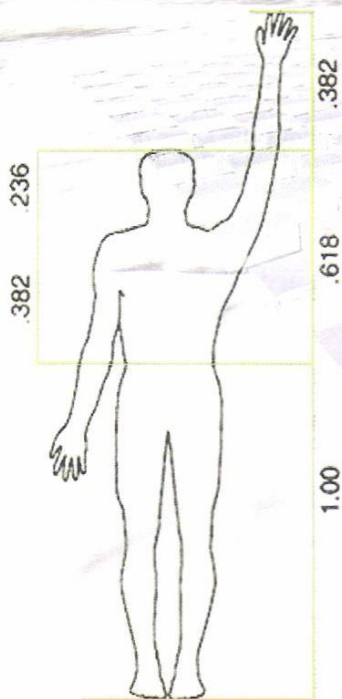


THE GOLDEN RATIO

Some irrational numbers are not only important for mathematicians. The golden ratio is an irrational number that is used in art and architecture and which has even been found in nature, for example to describe the proportions of the human face and body, the properties of some metals, and the position of the leaves on plants.

The golden ratio was first formally defined by the mathematician Euclid in 300 BC, although it was known by the followers of Pythagoras around 200 years before this time, and some people believe that it was also used by the Ancient Egyptians. Euclid defined the golden ratio in terms of a line segment that was divided into two unequal sections with lengths a and b . If the lengths satisfied the equality $(a + b) / a = a / b$ then the line segment was divided in the golden ratio. The actual value of the golden ratio is $(1 + \sqrt{5}) / 2$, which expands to 1.6180339887...

The Ancient Greeks found that buildings designed using the golden ratio were particularly pleasing to look at. The Parthenon in Athens is one example of the application of the golden ratio in Greek architecture. In 1509 an Italian priest, Luca Pacioli, published three books with the title *Divina Proportione*, which means 'the Divine Ratio'. He mentioned that the infinite, non-repeating nature of this number was like God being unique in the universe. Pacioli's book was illustrated by Leonardo da Vinci, who adopted the golden ratio in his paintings: for example, the proportions of the face of the Mona Lisa are based on the golden ratio. Today, artists, architects and designers continue to use the golden ratio in their work.



EXERCISES 1.4

A. Irrational Numbers

1. Indicate whether each expression is a rational or irrational number.

- a. $\sqrt{2}$ b. $\sqrt{4} + 1$
 c. $\sqrt{3} + \sqrt{9}$ d. $\sqrt{1} + \sqrt{16}$
 e. $\sqrt{\frac{25}{9}}$ f. $\sqrt{2.25}$

2. $A = \{-2, -1, -0.5, 2, \sqrt{2}, \sqrt{3}, \frac{3}{5}, \sqrt{5}, \sqrt{9}, \pi, 17, \frac{\sqrt{25}}{4}\}$

is given. List the elements of A which are included in each number set.

- a. integers
 b. rational numbers
 c. irrational numbers

B. Real Numbers

3. Graph each interval and write it as an inequality.

- a. $[-3, 3]$ b. $(-2, 2]$ c. $(2, \infty)$
 d. $[3, \infty)$ e. $(-\infty, -2)$ f. $(-\infty, -12]$

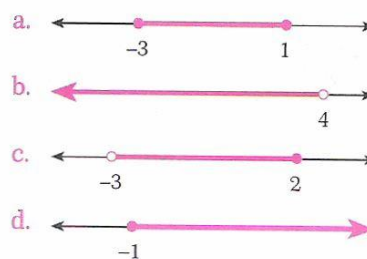
4. Graph each inequality and write it as an interval.

- a. $-2 < x < 3$ b. $-0.5 < x \leq 0$
 c. $x \geq -7$ d. $x < 100$

5. Write each interval as one interval if possible and graph the result.

- a. $[-2, 3) \cap (1, 5)$ b. $[2, 3) \cup (1, 5)$
 c. $(-\infty, 2) \cup (-3, \infty)$ d. $(-\infty, -1] \cap [1, \infty)$

6. Write each graph as an interval and as an inequality.



7. What can be said about the signs of the numbers x and y in each case?

- a. $xy > 0$ b. $xy < 0$ c. $\frac{x}{y} > 0$

- d. $\frac{x}{y} < 0$ e. $\frac{xy}{z^2} < 0$

8. If $z < 0$, what can be said about the signs of the numbers x and y in each case?

- a. $xyz > 0$ b. $xy^2z < 0$

- c. $\frac{xy}{z} < 0$ d. $\frac{x^2y^3}{z^3} < 0$

- e. $x + y < 0$ and $xy > 0$



9. $-2 < x \leq 0$ and $2 \leq y < 4$ are given. Find all possible values of each expression.

- a. $-2x$ b. $y - 3$
c. $3x - 0.5y$ d. $1.5x + 3y + 2$

10. x , y and z are negative real numbers such that

$$\frac{x}{0.3} = \frac{y}{0.5} = \frac{z}{0.45}. \text{ Order } x, y \text{ and } z.$$

11. x , y and z are positive real numbers such that

$$\frac{0.4}{3x} = \frac{0.5}{4y} = \frac{0.6}{5z}. \text{ Order } x, y \text{ and } z.$$

12. a , b , c and d are negative real numbers such that

$$\frac{1}{a+b} > \frac{1}{b+c} > \frac{1}{a+c} > \frac{1}{a+d}. \text{ Order } a, b, c \text{ and } d.$$

13. a. Find the sign of a and b if $ab^2 < 0$.

b. Find the sign of abc^2 if $a > 0$, $b > 0$ and $c < 0$.

14. Find the sign of $\frac{3-a}{4+b}$ if $a < 0$ and $b > 0$.

15. Prove each inequality for any real number a .

- a. $(a+5)(a-6) < (a+4)(a-5)$
b. $a(a-8) > 2(a-13)$
c. $a^4 + 4 \geq 4a^2$

16. Prove that

$$(*) \quad (a^2b^2 + a^2c^2 + b^2c^2 + 3) \geq 2(ab + ac + bc).$$

C. Absolute Value

17. Evaluate the expressions.

- a. $|3-7| + ||0| + |-2+6||$
b. $|-1.2 - (-2.7)| \cdot |-2| \cdot |\pi|$

18. Given $a < 0 < b$, simplify $|a-b| + |a| - |b|$.

19. Given $a < b < 0 < c$, simplify

$$|a-b| - |c-b| - |c-a|.$$

20. Given $a < 0 < b$, simplify

$$|b-a| - |2a-b| + |-b| - |a|.$$

21. Given $x < -\frac{1}{2}$, simplify $\frac{|2x+1|+x}{|x|-1}$.

22. Given $\frac{1}{8} < a < \frac{1}{5}$, simplify $|13a-2| + |5a-1|$.

23. $x < 0$ and $(x+y) > 0$ are given. Simplify

$$|-x| - |x-y| + |-x-y|.$$

24. Given $a < 0 < b$, simplify $\frac{a}{|a|} + \frac{|b|}{|b|} + \frac{ab}{|ab|}$.

25. Given $0 < x < 1$, simplify $\frac{|x^2 - 1| - |1 - x|}{|x^2 - x|}$.

26. Given $x < y < z < 0$, simplify $|xy - yz| - |xz - yz| - |zy - xy| - |xy|$.

27. Evaluate $|0.2 - 0.\overline{2}| + |0.0\overline{2} - \frac{1}{4}|$.

28. Find the maximum value of $\frac{1}{3} + \frac{|a+b|}{|a|+|b|}$.

29. Find the value of $\frac{2x+5y}{3y+x}$ at which $|2x-6y|$ takes its minimum value.

Mixed Problems

30. Prove that $\sqrt{5}$ is an irrational number.

31. Prove that the product of a rational number and an irrational number is irrational.

32. a and b are two positive real numbers such that $2.5 < \left|\frac{a}{b} + \frac{b}{a}\right| < 6$ and $ab = 2$. Find the interval for $a + b$.

33. For which values of a is the sign of $6a - 3a^2$ negative?

34. $ab = x$, $ac = y$, $bc = z$ and $x, y, z \in \mathbb{R}^+$ are given. Find $a + b + c$ in terms of x, y and z .

35. a and b are real numbers. Prove that $\frac{a^4 + b^4}{2} \geq \left(\frac{a^2 + b^2}{2}\right)^2 \geq a^2 b^2$.

36. Use an expression in absolute value notation to describe each case.

- The distance between x and 0 is less than 20.
- The distance between x and 3 is no more than 1.
- The distance between x and -5 is at least 4.

37. The height h of each member of a certain population satisfies the inequality $\left|\frac{h-168.5}{2.8}\right| \leq 1$, where h is measured in centimeters. Determine the interval on the real number line in which the heights of the population lie.

38. Find the maximum value of the expression $\frac{8}{|x-2| + |x+6|}$.

39. Find the maximum value and minimum value of $|x-3| + |x-4| - |2x-10|$.

5

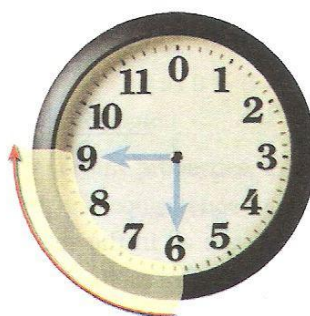
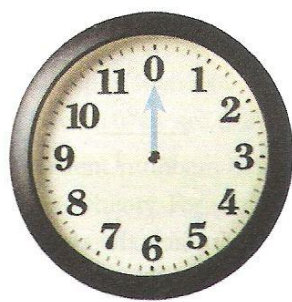
MODULAR ARITHMETIC (OPTIONAL)

Suppose it is 11 o'clock in the evening. If you have to be at school in ten hours' time, at what time should you be there? Or if today is Monday, what day of the week will it be twenty days later? Questions such as these are important in many different kinds of job. A special kind of arithmetic called **modular arithmetic** helps us to solve problems of this kind.

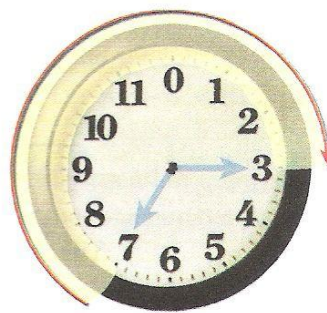
How can we begin to understand modular arithmetic? One way is to think about the hours on a clock. Imagine a clock face with numbers from zero (at the top) to 11, and one hand to show the hour (there is no minute hand). The hour hand can show one of twelve numbers in the finite set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

We can add numbers on the clock by moving the hour hand in a clockwise direction. For example, to add 6 and 3 on the clock, first we move the hour hand to 6, and then to add 3, we move the hour hand three more hours in a clockwise direction. The hand stops at 9, so $6 + 3 = 9$.

This is the same as regular addition. However, to find the sum $7 + 8$ on the clock, we move the hour hand to 7, and then move it clockwise through eight more hours. It stops at 3, so $7 + 8 = 3$.



plus 3 hours
 $6 + 3 = 9$



plus 8 hours
 $7 + 8 = 3$

Arithmetic with a clock like this is called **clock arithmetic**, and because there are twelve hours on the clock in our example we can say we have used **twelve-hour clock arithmetic**. In the twelve-hour clock arithmetic system, $8 + 9 = 5$, $11 + 1 = 0$ and $10 + 9 = 7$, etc.

Now that we have understood clock arithmetic, we can begin our study of modular arithmetic.

Remember that for $a, b, c, r \in \mathbb{Z}$ and $0 \leq r < b$, if $a = bc + r$ then r is called the remainder when a is divided by b .

For example,

$$29 = (7 \cdot 4) + 1, \text{ i.e. when } 29 \text{ is divided by } 7, \text{ the remainder is } 1.$$

$$\begin{array}{r|l} 29 & 7 \\ -28 & 4 \\ \hline 1 & \end{array}$$

$$-35 = [8 \cdot (-5)] + 5, \text{ i.e. when } -35 \text{ is divided by } 8, \text{ the remainder is } 5.$$

$$\begin{array}{r|l} -35 & 8 \\ -40 & -5 \\ \hline 5 & \end{array}$$

$$-3 = [4 \cdot (-1)] + 1, \text{ i.e. when } -3 \text{ is divided by } 4, \text{ the remainder is } 1.$$

$$\begin{array}{r|l} -3 & 4 \\ -4 & -1 \\ \hline 1 & \end{array}$$

Note

Remember: if $r = 0$ then $a = bc + r$ gives $a = bc + 0 = bc$. In this case, we say that a is a multiple of b .

Now look at the remainder obtained when some integers are divided by 4:

$$24 = 4 \cdot 6 + 0 \quad -17 = 4 \cdot (-5) + 3 \quad 33 = 4 \cdot 8 + 1 \quad -25 = 4 \cdot (-7) + 3$$

$$\begin{array}{r|l} 24 & 4 \\ -24 & 6 \\ \hline 0 & \end{array}$$

$$\begin{array}{r|l} -17 & 4 \\ -20 & -5 \\ \hline 3 & \end{array}$$

$$\begin{array}{r|l} 33 & 4 \\ -32 & 8 \\ \hline 1 & \end{array}$$

$$\begin{array}{r|l} -25 & 4 \\ -28 & -7 \\ \hline 3 & \end{array}$$

The remainders are all in the set $\{0, 1, 2, 3\}$. In fact, the remainder when any integer is divided by 4 is in the set $\{0, 1, 2, 3\}$. Notice that the set contains all the non-negative integers that are less than the divisor 4.

Similarly, the set of remainders when any integer is divided by 2 is $\{0, 1\}$, and the set of remainders when any integer is divided by 3 is $\{0, 1, 2\}$, and so on. We can generalize this result in the following way.

Result

Let m be a positive integer. Then the set of all remainders when any integer is divided by m is $\{0, 1, 2, 3, \dots, m-1\}$.



Of course, some numbers have the same remainder when they are divided by a particular integer. For example, $47 \div 5$ has remainder 2, and $12 \div 5$ also has remainder 2. This similarity helps us to define a new number relationship called **congruence**.

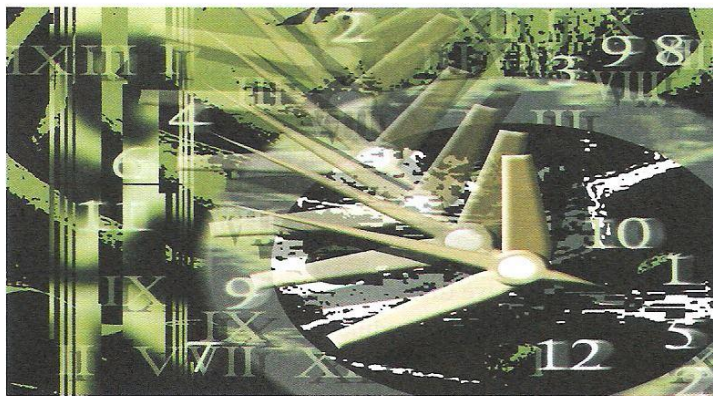
Definition

congruence, modulus

If two integers x and y have the property that $x - y$ is divisible by another integer m ($m \neq 1$) then x and y are called **congruent modulo m** . The number m is called the **modulus**. We say ' x is congruent to y (modulo m)' and write $x \equiv y \pmod{m}$.

For example, $47 \equiv 2 \pmod{5}$ and $47 \equiv 12 \pmod{5}$ since $47 - 2$ and $47 - 12$ are both divisible by 5.

Similarly, $-16 \equiv 8 \pmod{12}$ since $-16 - 8$ is divisible by 12.



In English, a.m. means 'in the morning' (before 12:00 in the day) and p.m. means 'in the afternoon / evening' (after 12:00 in the day). Midnight is 12 a.m. and midday is 12 p.m.

Twelve-hour clock arithmetic uses a modulus of 12. In arithmetic modulo 12 we write calculations such as $9 + 3 = 0$ or $3 - 5 = 10$ as $9 + 3 \equiv 0 \pmod{12}$ or $3 - 5 \equiv 10 \pmod{12}$ to show that we are using modular arithmetic.

Congruent numbers have some important properties, and are very useful in many areas of number theory. For example, we can use congruence to test whether one number is divisible by another number.

EXAMPLE 105

Find the possible values of m that satisfy each congruence.

- a. $13 \equiv 3 \pmod{m}$ b. $15 \equiv 4 \pmod{m}$

Solution

- a. By the definition of congruence, $13 - 3 = 10$ must be divisible by m .

So the possible values of m are the positive divisors of 10: $m \in \{2, 5, 10\}$. (Remember that by the definition, $m \neq 1$.)

- b. $15 - 4 = 11$ must be divisible by m . So m can only be 11.

EXAMPLE 106 Find the smallest non-negative integer that can be the result of each operation.

- a. $12 + 8 \pmod{7}$ b. $5 - 17 \pmod{6}$

Solution a. $12 + 8 = 20 \equiv 6 \pmod{7}$ since $20 - 6$ is divisible by 7. Note that $20 - 13$ is also divisible by 7, but we are looking for the smallest non-negative integer in modulo 7, so the answer must be between zero and 6.

- b. $5 - 17 = -12 \equiv 0 \pmod{6}$ since $-12 - 0$ is divisible by 6.

There is a general strategy for solving problems such as these: divide the result of the regular operation by the modulus and write the remainder as the result. For example, 20 divided by 7 gives 6 as remainder and -12 divided by 6 gives zero as remainder, so 6 and zero are the smallest non-negative integers.

EXAMPLE 107 If today is Wednesday, what day of the week will it be in 17 days' time?

Solution We can begin problems such as these by giving each day of the week a number, as follows:

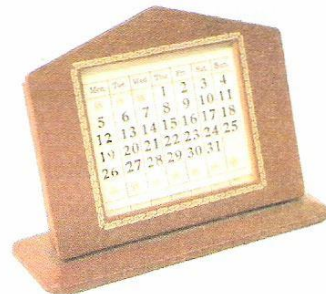
Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
0	1	2	3	4	5	6

Since we have the numbers 0, 1, 2, 3, 4, 5 and 6, we will do our arithmetic in modulo 7.

Today is **Wednesday** (day 3). To find the day which is 17 days later, we can write

$$3 + 17 = 20 \equiv 6 \pmod{7}.$$

So the answer is the **sixth** day, which is Saturday.



Check Yourself 18

- Perform the operations in twelve-hour clock arithmetic.
 - $3 + 5$
 - $6 + 8$
 - $3 \cdot 5$
 - $(3 \cdot 4) - 13$
- How many possible remainders are there when an integer is divided by 11?
- Find all possible values of m if $13 \equiv 4 \pmod{m}$

Answers

1. a. 8 b. 2 c. 3 d. 11 2. 11 3. $m \in \{3, 9\}$



Theorem

Let $a, b, c, d, m \in \mathbb{Z}$ and $m \neq 1$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then the following statements are true.

1. $a + c \equiv b + d \pmod{m}$
2. $ac \equiv bd \pmod{m}$
3. $a^n \equiv b^n \pmod{m}$

For example, since $35 \equiv 3 \pmod{8}$ and $21 \equiv 5 \pmod{8}$ then this theorem tells us

$$35 + 21 = 56 \equiv 0 \pmod{8} \text{ and } 3 + 5 = 8 \equiv 0 \pmod{8},$$

$$35 \cdot 21 = 735 \equiv 7 \pmod{8} \text{ and } 3 \cdot 5 = 15 \equiv 7 \pmod{8}, \text{ and}$$

$$35^3 \equiv 3^3 \pmod{8} \text{ (check it!).}$$

Proof 1. If $a \equiv b \pmod{m}$ then by the definition of congruence there exists $k_1 \in \mathbb{Z}$ such that $a - b = mk_1$. If $c \equiv d \pmod{m}$ then there exists $k_2 \in \mathbb{Z}$ such that $c - d = mk_2$.

Adding these two equations side by side gives

$$(a - b) + (c - d) = mk_1 + mk_2$$

$$(a + c) - (b + d) = m(k_1 + k_2).$$

But $k_1 + k_2 \in \mathbb{Z}$, and so $(a + c) - (b + d)$ is a multiple of m .

We can conclude that $(a + c) \equiv (b + d) \pmod{m}$.

2. As above, we begin by writing $a = b + mk_1$ and $c = d + mk_2$. So

$$ac = (b + mk_1)(d + mk_2) = bd + mbk_2 + mdk_1 + m^2k_1k_2 = bd + m(bk_2 + dk_1 + mk_1k_2), \text{ i.e.}$$

$$ac - bd = m(bk_2 + dk_1 + mk_1k_2).$$

Since the expression in brackets has an integer value, $ac - bd$ is a multiple of m . Therefore, $ac \equiv bd \pmod{m}$.

3. This proof is a generalization of part 2. It is left as an exercise for you.

Note

As a consequence of the above theorem, if $a \equiv b \pmod{m}$ and $p \in \mathbb{Z}$ then

1. $(a + p) \equiv (b + p) \pmod{m}$.
2. $ap \equiv bp \pmod{m}$.

EXAMPLE 108 Solve the equations.

a. $x + 3 \equiv 4 \pmod{8}$

b. $2x \equiv 1 \pmod{7}$

c. $3x + 2 \equiv 4 \pmod{8}$

Solution a. $x + 3 - 3 \equiv 4 - 3 \pmod{8}$ b. $4 \cdot 2x \equiv 4 \cdot 1 \pmod{7}$ c. $3x + 2 + 6 \equiv 4 + 6 \pmod{8}$

$$x \equiv 1 \pmod{8}$$

$$8x \equiv 4 \pmod{7}$$

$$3x \equiv 2 \pmod{8}$$

$$\text{or } x = 8k + 1, k \in \mathbb{Z}$$

$$1x \equiv 4 \pmod{7}$$

$$3 \cdot 3x \equiv 3 \cdot 2 \pmod{8}$$

$$x \equiv 4 \pmod{7}$$

$$x \equiv 6 \pmod{8}$$

$$\text{or } x = 7k + 4, k \in \mathbb{Z}$$

$$\text{or } x = 8k + 6, k \in \mathbb{Z}$$



EXAMPLE 109

- a. Find the remainder when 19^{53} is divided by 8.
 b. Find the remainder when 2222^{5555} is divided by 7.
 c. Find the remainder when $(38 \cdot 4^{48}) + (14 \cdot 15^{78})$ is divided by 9.

Solution a. When 19 is divided by 8 the remainder is 3, so $19 \equiv 3 \pmod{8}$.

By part 3 of the previous theorem we get $19^{53} \equiv 3^{53} \pmod{8}$.

This equation means that the remainder when 19^{53} is divided by 8 is the same as the remainder when 3^{53} is divided by 8. So we need to find the smallest non-negative integer that is equivalent to 3^{53} in modulo 8. To do this we proceed as follows.

$$3^1 \equiv 3 \pmod{8}$$

$$3^2 = 9 \equiv 1 \pmod{8}$$

Now we have an important result. When an expression is congruent to 1, any power of that expression will also be congruent to 1.

$$(3^2)^{26} \equiv 1^{26} \pmod{8}$$

$$3^{52} \equiv 1 \pmod{8}$$

$$3 \cdot 3^{52} \equiv 3 \cdot 1 \pmod{8}$$

$$3^{53} \equiv 3 \pmod{8}$$

Since $3^{53} \equiv 19^{53} \equiv 3 \pmod{8}$, the remainder is 3.

- b. We can evaluate 2222^{5555} in modulo 7 as follows.

$$2222 \equiv 3 \pmod{7}$$

$$2222^2 \equiv 3^2 \equiv 2 \pmod{7}$$

$$(2222^2)^3 \equiv 2^3 \equiv 1 \pmod{7}$$

$$(2222^6)^{925} \equiv 1^{925} \pmod{7}$$

$$2222^{5550} \equiv 1 \pmod{7}$$

$$2222^{5555} \equiv 1 \cdot 5 \equiv 5 \pmod{7} \quad (\text{note that } 2222^{5555} = 2222^{5550} \cdot 2222^5 \text{ and } 2222^5 \equiv 5)$$

- c. We can find the remainder of the sum or product of two numbers by first finding the remainder of each number and then substituting these remainders in the place of the numbers.

Note that $38 \equiv 2 \pmod{9}$ and $14 \equiv 5 \pmod{9}$.

We can evaluate 4^{48} and 15^{78} separately as follows.

$$4 \equiv 4 \pmod{9} \qquad 15 \equiv 6 \pmod{9}$$

$$4^2 = 16 \equiv 7 \pmod{9} \qquad 6^2 \equiv 0 \pmod{9}$$

$$4^3 = 4^2 \cdot 4 \equiv 7 \cdot 4 \equiv 1 \pmod{9} \qquad (6^3)^{39} \equiv 0 \pmod{9}$$

$$(4^3)^{16} \equiv 1 \pmod{9}$$

By substituting these values into the original expression, we obtain

$$(38 \cdot 4^{48}) + (14 \cdot 15^{78}) \equiv (2 \cdot 1) + (5 \cdot 0) \equiv 2 \pmod{9}. \text{ So the remainder is 2.}$$



EXAMPLE 110 Find the last digit of the number 83^{95} .

Solution The problem of finding the last digit of a number is the same as the problem of finding the remainder when the number is divided by 10, i.e. its value in modulo 10.

$$83 \equiv 3 \pmod{10}$$

$$83^2 \equiv 3^2 \equiv 9 \pmod{10}$$

$$83^3 \equiv 9 \cdot 3 \equiv 7 \pmod{10}$$

$$83^4 \equiv 7 \cdot 3 \equiv 1 \pmod{10}$$

$$83^{95} = (83^4)^{23} \cdot 83^3 \equiv 1^{23} \cdot 7 \equiv 7 \pmod{10}$$

So the last digit of 83^{95} is 7.

EXAMPLE 111 Find the remainder when 65^{6n-1} is divided by 9 ($n \in \mathbb{N}$).

Solution Let us calculate the first few powers of 65 in modulo 9.

$$65 \equiv 2 \pmod{9}$$

$$65^2 \equiv 4 \pmod{9}$$

$$65^3 \equiv 8 \pmod{9}$$

$$65^5 \equiv 4 \cdot 8 \equiv 5 \pmod{9}$$

$$65^6 \equiv 1 \pmod{9}$$

$$(65^6)^{n-1} = 65^{6n-6} \equiv 1^{6n-6} \equiv 1 \pmod{9}$$

$$65^{6n-6} \cdot 65^5 \equiv 1 \cdot 5 \pmod{9}$$

$$65^{6n-1} \equiv 5 \pmod{9}$$

So 65^{6n-1} divided by 9 has remainder 5.

Check Yourself 19

1. Solve $x - 5 \equiv 2 \pmod{9}$.
2. Find the remainder when 3^{100} is divided by 7.
3. Find the remainder when $(13^{100} \cdot 23) + (6 \cdot 7^{2007})$ is divided by 8.
4. Find the last digit of 13^{1003} .
5. Find the remainder when 6^{n-5} is divided by 10 ($n \in \mathbb{N}$, $n > 5$).

Answers

1. $9k + 7$, $k \in \mathbb{Z}$
2. 4
3. 1
4. 7
5. 6

EXAMPLE 112 Prove that $100^n - 1$ is divisible by 99 ($n \in \mathbb{N}$).

Solution Notice that $99 = 11 \cdot 9$, and 11 and 9 are both prime. So if $100^n - 1$ is divisible by both 11 and 9, then by the divisibility rules it is divisible by 99.

So we must show that $100^n - 1 \equiv 0 \pmod{11}$ and $100^n - 1 \equiv 0 \pmod{9}$.

$$100 \equiv 1 \pmod{11}$$

$$100 \equiv 1 \pmod{9}$$

$$100^n \equiv 1 \pmod{11}$$

$$100^n \equiv 1 \pmod{9}$$

$$100^n - 1 \equiv 0 \pmod{11} \quad (1)$$

$$100^n - 1 \equiv 0 \pmod{9} \quad (2)$$

By (1) and (2), $100^n - 1$ is divisible by 99.

EXAMPLE 113 Prove the rule for divisibility by 9: if a number is divisible by 9 then the sum of its digits must be a multiple of 9.

Solution Let $x = a_k \dots a_2 a_1 a_0$ be any integer such that $a_0, a_1, a_2, \dots, a_k$ are its digits.

Let us try to show that the number $x = (a_0 \cdot 10^0) + (a_1 \cdot 10^1) + (a_2 \cdot 10^2) + \dots + (a_k \cdot 10^k)$ and the sum of its digits $y = a_k + \dots + a_2 + a_1 + a_0$ are congruent in modulo 9.

Note that

$$a_0 \cdot 10^0 \equiv a_0 \pmod{9}$$

$$\text{since } 1 \equiv 1 \pmod{9}$$

$$a_1 \cdot 10^1 \equiv a_1 \pmod{9}$$

$$\text{since } 10 \equiv 1 \pmod{9}$$

$$a_2 \cdot 10^2 \equiv a_2 \pmod{9}$$

$$\text{since } 100 \equiv 1 \pmod{9}$$

$\vdots$

$$a_k \cdot 10^k \equiv a_k \pmod{9}$$

$$\text{since } 10^k \equiv 1 \pmod{9}, k \in \mathbb{N}.$$

Adding the above congruences side by side we get

$$(a_0 \cdot 10^0) + (a_1 \cdot 10^1) + (a_2 \cdot 10^2) + \dots + (a_k \cdot 10^k) \equiv a_0 + a_1 + a_2 + \dots + a_k \pmod{9}, \text{ or } x \equiv y \pmod{9}.$$

This means that the remainder when a number is divided by 9 is equal to the remainder when the sum of its digits is divided by 9. Therefore, if the sum of the digits of a number is divisible by 9 (that is, if it has remainder zero when divided by 9), then the number itself is also divisible by 9 (that is, it has remainder zero when divided by 9).

EXAMPLE 114 Find the smallest non-negative integer x which satisfies $1^3 + 2^3 + 3^3 + \dots + 36^3 \equiv x \pmod{37}$.

Solution Two integers are congruent modulo m if their difference is a multiple of m .

$$\text{So } 36 \equiv -1 \pmod{37}$$

$$\text{On the other hand, } 18 \equiv 18 \pmod{37}$$

$$35 \equiv -2 \pmod{37}$$

$$17 \equiv 17 \pmod{37}$$

$\vdots$

$\vdots$

$$20 \equiv -17 \pmod{37}$$

$$2 \equiv 2 \pmod{37}$$

$$19 \equiv -18 \pmod{37}.$$

$$1 \equiv 1 \pmod{37}.$$

$$\text{Therefore, } 1^3 + 2^3 + \dots + 17^3 + 18^3 + (-18)^3 + (-17)^3 + \dots + (-2)^3 + (-1)^3 \equiv 0 \pmod{37}.$$

Consequently, $x = 0$.

Theorem**Fermat's Little Theorem**

Let p be a prime number and let a be an integer which is not divisible by p . Then $a^{p-1} \equiv 1 \pmod{p}$.

For example, 5 is a prime number and 12 is not divisible by 5, so $12^{5-1} \equiv 1 \pmod{5}$.

Similarly, $8^{13-1} \equiv 1 \pmod{13}$ and $24^{17-1} \equiv 1 \pmod{17}$.

EXAMPLE**115**

$M = (8^{39} + 6^{48} + 15^{48} + 20^{39})^{200!}$ is given. Find the smallest non-negative value of x which satisfies $M \equiv x \pmod{7}$.

Solution

Since $8 \equiv 1$, $6 \equiv -1$, $15 \equiv 1$ and $20 \equiv -1$ in modulo 7, we can write

$$\begin{aligned} M &= (8^{39} + 6^{48} + 15^{48} + 20^{39})^{200!} \equiv (1^{39} + (-1)^{48} + 1^{48} + (-1)^{39})^{200!} \pmod{7} \\ &\equiv (1 + 1 + 1 - 1)^{200!} \pmod{7} \\ &\equiv 2^{200!} \pmod{7}. \end{aligned}$$

Note that $200! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \dots \cdot 200 = 6 \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \dots \cdot 200)$, so

$$M \equiv (2^6)^{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \dots \cdot 200} \pmod{7}.$$

7 is a prime number and 2 is not divisible by 7, so by Fermat's Little Theorem, $2^6 \equiv 1 \pmod{7}$. Consequently, $M \equiv 1^{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \dots \cdot 200} \equiv 1 \pmod{7}$.

So the smallest positive value of x is 1.

EXAMPLE**116**

Find the remainder left when the 90-digit number 999...9 is divided by 89.

Solution

We need to evaluate 99...9 in modulo 89.

$$\text{Let } 999\dots9 \equiv x \pmod{89}$$

$$99\dots9 + 1 \equiv x + 1 \pmod{89}$$

$$10^{90} \equiv x + 1 \pmod{89}.$$

Note that 89 is a prime number and 10 is not divisible by 89. Therefore, by Fermat's Little Theorem,

$$10^{89-1} = 10^{88} \equiv 1 \pmod{89}$$

$$10^{88} \cdot 10^2 = 10^{90} \equiv 11 \pmod{89} \quad (\text{since } 10^2 \equiv 11 \pmod{89})$$

$$11 \equiv x + 1 \pmod{89}$$

$$10 \equiv x \pmod{89}.$$

So the remainder is 10.

Check Yourself 20

- Find the remainder when 8^{103} is divided by 13.
- Prove that $20^{15} - 1$ is divisible by 341.

Answers

- 5
- Hint: Consider the number in modulo 11 and modulo 31.

ISBN NUMBERS

ISBN is the abbreviation for International Standard Book Number. Each edition of a published book receives its own ISBN. An ISBN is either 10 or 13 digits long, and consists of four or five blocks of digits. If the ISBN is 13 digits long then the first block of digits is 978 or 979. The second block of digits identifies the country where the book was published or its language (for example, 0 or 1 for an English-speaking country, 2 for French-speaking countries, etc.) The third block of digits identifies the publisher. The fourth block of digits is the number assigned to the book by the publishing company. The final block is a number called the check digit.

Consider the ten-digit ISBN 975-266-088-6. To calculate the check digit for this ISBN, follow the steps.

Step 1: Multiply the nine digits before the check digit by weighted values and find their sum. The weighted values are 1 for the first digit from the left, 2 for the second digit, 3 for the third digit, etc. In our example, the result is

$$1 \times 9 + 2 \times 7 + 3 \times 5 + 4 \times 2 + 5 \times 6 + 6 \times 6 + 7 \times 0 + 8 \times 8 + 9 \times 8 = 248.$$

Step 2: Use modulo 11 to determine the check digit:

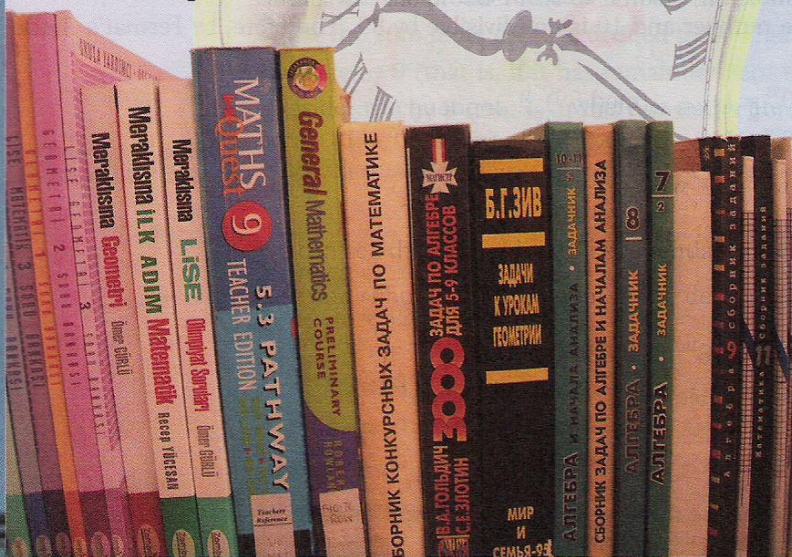
$$248 \equiv 6 \pmod{11}, \text{ so the check digit is 6.}$$

To determine whether a check digit in an ISBN number is valid, we multiply all ten digits by weighted values to get a value x and check that $x \equiv 0 \pmod{11}$. In our case,

$$1 \times 9 + 2 \times 7 + 3 \times 5 + 4 \times 2 + 5 \times 6 + 6 \times 6 + 7 \times 0 + 8 \times 8 + 9 \times 8 + 10 \times 6 = 308 \equiv 0 \pmod{11}, \text{ so the check digit is valid.}$$



The two most common errors which occur when somebody writes down or types an ISBN number are an altered digit or the transposition of adjacent digits. Since 11 is a prime number, the ISBN check digit method ensures that these two kinds of error will always be detected. Can you see why? Can you think of any other situations in which a check digit might be useful? What would be an appropriate modulus for these situations?



EXERCISES 1.5

1. Perform the operations. Find the quotient q and the remainder r in each case.

$$\begin{array}{lll} \text{a. } -24 \overline{) 9} & \text{b. } 24 \overline{) 5} & \text{c. } -89 \overline{) 9} \\ \text{d. } 111 \overline{) 11} & \text{e. } -1001 \overline{) 10} & \text{f. } -1001 \overline{) 101} \end{array}$$

2. Determine whether each statement is true or false.

$$\begin{array}{ll} \text{a. } 2038 \equiv 4 \pmod{8} & \text{b. } 7679 \equiv -26 \pmod{5} \\ \text{c. } -38 \equiv -7 \pmod{9} & \text{d. } -1824 \equiv -44 \pmod{10} \\ \text{e. } -1001 \equiv 1 \pmod{5} & \text{f. } -12 \equiv -12 \pmod{7} \\ \text{g. } 16 \equiv -2 \pmod{6} & \text{h. } -8 \equiv 8 \pmod{3} \end{array}$$

3. Find the smallest non-negative integer value of x which satisfies each congruence.

$$\begin{array}{l} \text{a. } 34 \equiv x \pmod{8} \\ \text{b. } -47 \equiv x \pmod{3} \\ \text{c. } -x \equiv 4 \pmod{5} \\ \text{d. } 3 - x \equiv -2 \pmod{6} \\ \text{e. } -83 \equiv 4 - x \pmod{11} \\ \text{f. } 6 - x \equiv x - 4 \pmod{7} \\ \text{g. } 23 \equiv 6 + x \pmod{9} \\ \text{h. } 34 \equiv 2x + 5 \pmod{6} \\ \text{i. } 87 + 3x \equiv -2x + 24 \pmod{4} \\ \text{j. } 123 - 4x \equiv 32 - x \pmod{9} \end{array}$$

4. Find all possible values of m if

$$\text{a. } 20 \equiv 8 \pmod{m} \quad \text{b. } -7 \equiv -22 \pmod{m}$$

5. Find the remainder when each expression is divided by 3.

$$\begin{array}{ll} \text{a. } 8^{92} & \text{b. } 14^{25} + 274^{12} \\ \text{c. } (13^{15} \cdot 5^{26}) + (4 \cdot 26^{32}) & \\ \text{d. } (42^{34} \cdot 16^8) + (25^{37} \cdot 82^5) & \end{array}$$

6. Find the last digit of each number.

$$\begin{array}{ll} \text{a. } 15^{92} & \text{b. } 83^{34} + 16^{14} \\ \text{c. } 48^{27} \cdot 23^{16} & \text{d. } -26^{32} \cdot 100 + 138^5 \cdot 44 \\ \text{e. } 153^{266} & \text{f. } 128^{152} \\ \text{g. } 7^{2005} & \end{array}$$

7. Find the remainder when

$$\begin{array}{l} \text{a. } 13^{146} + 7^{95} \text{ is divided by } 5. \\ \text{b. } 15^{96} + 14^{24} \text{ is divided by } 6. \\ \text{c. } 18^{79} \cdot 24^{145} \text{ is divided by } 7. \\ \text{d. } 47^{136} + 16^{83} \text{ is divided by } 11. \end{array}$$

8. Find the remainder when each expression is divided by 7. ($n \in \mathbb{N}$)

a. 3^{6n+12}

b. 6^{6n+4}

c. 4^{18n+8}

d. $(-2)^{42n+32}$

e. $(-3)^{36n-12}$

f. 5^{48n+3}

g. 2^{75n-18}

9. a and b are two integers such that a leaves remainder 3 when it is divided by 7 and b leaves remainder 5 when it is divided by 7. Find the remainder when each expression is divided by 7.

a. ab

b. a^2

c. $3a - 26b$

d. $a^6 + 32b^3$

e. $a^{23} - 45b^{16}$

10. Which day of the week will it be

a. 56 days after a Saturday?

b. 264 days after a Friday?

c. 312 days after a Wednesday?

11. Find the smallest non-negative integer value of x which satisfies each congruence.

a. $16^{1991} \equiv x \pmod{7}$

b. $1991^{92} \equiv x \pmod{5}$

c. $1993^x \equiv 2 \pmod{5}$

d. $2007^x \equiv 1 \pmod{10}$

12. Write the solution set for each congruence.

a. $x + 2 \equiv 4 \pmod{6}$

b. $x + 3 \equiv 5 \pmod{7}$

c. $2x + 5 \equiv 3 \pmod{4}$

13. Solve $3x \equiv 1 \pmod{23}$.

14. Solve $2^{10} \cdot 4^{20} \cdot 5^{50} \equiv x \pmod{7}$.

15. Solve $32 \equiv 100x \pmod{7}$.

16. Solve $x^2 \equiv 1 \pmod{6}$.

17. $x^2 + 7x \equiv 15 \pmod{x}$ is given. Find the sum of the possible values of x .

18. Find the smallest non-negative value of x which satisfies $7^{1962} \equiv x \pmod{11}$.

19. Let $M = 0! + 2! + 4! + \dots + 88!$

where $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$.

Find the smallest non-negative value of x which satisfies $M^{78} \equiv x \pmod{7}$.



20. Find the remainder left when

★ $1^n + 2^{2n+1} + 3^n + 4^n + 5^{2n}$ is divided by 6. ($n \in \mathbb{N}$)

21. Find the remainder when $3^{3n} + 5^{6n+1} + 6^{3n+2}$ is divided by 9. ($n \in \mathbb{N}$)

22. $x \equiv 2 \pmod{7}$, $x \equiv 3 \pmod{5}$ and $x \equiv 4 \pmod{6}$ are given. Find the smallest possible non-negative value of x .

23. Find the last digit of the number $(1! + 2! + 3! + \dots + 2004!)^{200}$.

24. Let $a = \underbrace{66666\dots6}_{24 \text{ digits}}$ and $b = \underbrace{8888\dots8}_{48 \text{ digits}}$.

Find the smallest non-negative value of x which satisfies $x \equiv ab \pmod{9}$.

25. Find the smallest three-digit number x which satisfies $14^x + 8^x \equiv 0 \pmod{5}$.

26. Prove that for $n \in \mathbb{N}$, $3^{6n} - 2^{6n}$ is divisible by 35.

27. Prove that for $n \in \mathbb{N}$, if $a \equiv b \pmod{m}$ then $a^n \equiv b^n \pmod{m}$.

28. Show that for any integer a , either $a^2 + a + 1 \equiv 1 \pmod{3}$ or $a^2 + a + 1 \equiv 0 \pmod{3}$.

29. Prove that for $n \in \mathbb{Z}$, 504 divides $n^9 - n^3$.

30. Using modular arithmetic, prove the rules of divisibility by 3, 4, 5 and 11.

31. Prove that $(2 \cdot 4 \cdot 6 \dots 40) \cdot (3 \cdot 5 \cdot 7 \dots 39) + 1$ is divisible by 41.

CHAPTER SUMMARY

1. Natural Numbers

- The set of **natural numbers** is $\mathbb{N} = \{1, 2, 3, \dots, n, n+1, \dots\}$.
- The set of **whole numbers** is $W = \{0, 1, 2, 3, \dots, n, n+1, \dots\}$.
- The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are called **digits**. A sequence of one or more digits makes a number. 103 is a number with three digits: 1, 0 and 3.
- A number that is divisible by 2 is **even**, otherwise it is **odd**.
- The **division algorithm**: $a = (b \cdot c) + r$ for $0 \leq r < b$. The number r is called the **remainder**. When the remainder is zero we say that a is **divisible** by b .
- If a is divisible by b then b is called a **factor** (or **divisor**) of a , and a is called a **multiple** of b .
- A **prime number** is a natural number which has only two factors, 1 and the number itself. A natural number that has more than two factors is called a **composite number**.
- A natural number is divisible
 - by 2 if it is even.
 - by 3 if the sum of its digits is a multiple of 3.
 - by 4 if the number formed by the last two digits is divisible by 4.
 - by 5 if its last digit is 0 or 5.
 - by 8 if the number formed by the last three digits is divisible by 8.
 - by 9 if the sum of its digits is a multiple of 9.
 - by 11 if the difference between the sum of the odd-numbered digits and the sum of the even-numbered digits is a multiple of 11.
- Let x and y be two natural numbers. If the remainder when x is divided by a is m , and the remainder when y is divided by a is n , then
 - the remainder when $x + y$ is divided by a is the same as the remainder when $m + n$ is divided by a .
 - the remainder when xy is divided by a is the same as the remainder when mn is divided by a .
- The **greatest common factor** of two natural numbers a and b is the largest natural number which divides both a and b . It is denoted by $\text{GCF}(a, b)$ or simply (a, b) .

- The **least common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b . It is denoted by $\text{LCM}(a, b)$ or simply $[a, b]$.
- $\text{GCF}(a, b) \cdot \text{LCM}(a, b) = ab$
- Two natural numbers a and b are called **relatively prime numbers** if their greatest common factor is 1.
- Let p and q be two relatively prime numbers and $c \in \mathbb{N}$. If $p|c$ and $q|c$ then $pq|c$.

2. Integers

- The set of **integers** is $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
- The set of **negative integers** is $\mathbb{Z}^- = \{\dots, -4, -3, -2, -1\}$ and the set of **positive integers** is $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$.
- $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$
- Order of Operations (BEDMAS Rule)**
 - Brackets,
 - Exponents,
 - Division and Multiplication,
 - Addition and Subtraction

3. Rational Numbers

- A number which can be expressed as the ratio of an integer to a non-zero integer is called a **rational number**. The set of rational numbers is $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$. The number $\frac{a}{b}$ is called a **fraction**. a is called the **numerator** and b is called the **denominator** of the fraction.
- If we divide the numerator of a fraction by the denominator then we can convert a rational number to **decimal form**. In a decimal number $a.b$, a is called the **whole part**, b is called the **decimal part** and the period (.) is called the **decimal point**. Every rational number can be written as a decimal number which either **repeats** or **terminates**.
- Operations with Fractions**

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$
- There are infinitely many rational numbers between two different rational numbers. This is called the **density property** of rational numbers.



4. Real Numbers

- Irrational numbers are numbers that cannot be expressed as the ratio of two integers. The set of irrational numbers is denoted by $\mathbb{Q}'$.
- The set of **real numbers** is $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
- The **real number line** is a line on which every point corresponds to a real number. Conversely, every real number corresponds to a point on the line. On the real number line, the point with coordinate zero is called the **origin**. All numbers to the right of zero are called **positive real numbers**, and all numbers to the left of zero are called **negative real numbers**.
- Let a and b be two numbers on the real number line.
 - If a lies to the left of b then $a < b$.
 - If a lies to the right of b then $a > b$.
- For any real number x , the **absolute value** of x is denoted by $|x|$ and $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.
- The absolute value of a number is never negative.
- For any real numbers a and b ,
 - $|a - b| = |b - a|$.
 - $|ab| = |a| \cdot |b|$ and $|a^n| = |a|^n$.
 - $|a + b| \leq |a| + |b|$.

5. Modular Arithmetic

- Let m be a positive integer. The set of all possible remainders when any integer is divided by m is $\{0, 1, 2, 3, \dots, m-1\}$.
- If two integers x and y have the property that $x - y$ is divisible by another integer m ($m \neq 1$) then x and y are said to be **congruent modulo m** . The number m is called the **modulus**, and the statement ' x is congruent to y in modulo m ' is denoted by $x \equiv y \pmod{m}$.
- Let $a, b, c, d, m \in \mathbb{Z}$ and $m \neq 1$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then the following statements are true.
 - $a + c \equiv b + d \pmod{m}$
 - $ac \equiv bd \pmod{m}$
 - $a^n \equiv b^n \pmod{m}$
- **Fermat's Little Theorem:** If p is a prime number and a is an integer not divisible by p then $a^{p-1} \equiv 1 \pmod{p}$.

Concept Check

1. What is the difference between a set of natural numbers and a set of digits?
2. Is the set of natural numbers enough for counting?
3. Suppose that we do not know the digit 0. How can we express the number 10^{98} ?
4. Is 1 a prime number? Explain your answer.
5. How many prime numbers are there?
6. State the divisibility rules for 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.
7. Can the least common multiple of two natural numbers be divisible by the greatest common factor of these numbers?
8. How many common factors do a prime number p and the composite number p^2 have?
9. Can a prime number be negative?
10. Can we find an integer which is neither negative nor positive?
11. Is the equality $(-7)^2 = -(7^2)$ true? What about $(-7)^3 = -(7^3)$?
12. Is it true that all integers are rational numbers?
13. How can we find a rational number between any two rational numbers?
14. Let $x, y \in \mathbb{Q}$ such that $0 < x < y < 1$. Which one is greater: the ratio of x to y or the ratio of y to x ?
15. How can we express a terminating decimal as a fraction? What about a repeating decimal?
16. Is it true that all repeating decimals are irrational numbers?
17. Is there a number which is both rational and irrational? Explain your answer.
18. When we compare two real numbers, how many possible cases do we have?
19. Give examples of open, closed and half-open intervals. Express them in inequality notation.
20. Can a real number have a negative absolute value?
21. Explain why $-x$ could be positive.
22. Suppose that the sum of the absolute values of two real numbers is zero. What are these numbers?
23. Suppose n and m are natural numbers such that $n > m$. How many possible remainders are there when we divide n by m ?
24. What is the relation between a clock and modular arithmetic?
25. Express the statement 'when a is divided by m , the remainder is b ' as a congruence in modular notation.

CHAPTER REVIEW TEST 1A

1. Evaluate $3^2 - 2^5 + 3^4 - 3^2 + 2^3$.

- A) 49 B) 57 C) 66 D) 75 E) 81

2. Find the number of digits in $20^4 \cdot 40^6 \cdot 50^8$.

- A) 26 B) 27 C) 28 D) 29 E) 30

3. Evaluate $\frac{\frac{2}{5} - 1}{1 - \frac{3}{2 - \frac{1}{5}}} + \frac{9}{10}$.

- A) $\frac{1}{2}$ B) 1 C) 2 D) 4 E) 8

4. How many numbers are there between 1 and 620 which are divisible by both 3 and 4?

- A) 49 B) 50 C) 51 D) 52 E) 53

5. For $x, y \in \mathbb{N}$, which one of the following is always an odd number?

$$(n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1)$$

- A) $(x+y)!$ B) 2^x C) $x^2 + 1$
D) $3x + 5y$ E) $5! \cdot x + 1$

6. $A = \underbrace{55\dots5}_{26 \text{ digits}}$ and $B = \underbrace{77\dots7}_{50 \text{ digits}}$ are given. Find the remainder when $B - A$ is divided by 9.

- A) 4 B) 5 C) 6 D) 7 E) 8

7. a and b are two consecutive odd numbers such that $\text{LCM}(a, b) - \text{GCF}(a, b) = 98$. Find $a + b$.

- A) 16 B) 20 C) 24 D) 28 E) 30

8. Evaluate

$$(2 + 4 + 6 + \dots + 2004) - (3 + 5 + 7 + \dots + 2003).$$

- A) 2003 B) 1553 C) 1253
D) 1003 E) 1903



9. Which one of the following numbers is the smallest?

- A) $-\frac{3}{2}$ B) $-\frac{25}{24}$ C) $-\frac{12}{9}$
 D) $-\frac{17}{9}$ E) $-\frac{50}{27}$

10. Which one of the following numbers is between $\frac{11}{13}$ and $\frac{13}{15}$?

- A) $\frac{9}{11}$ B) $\frac{9}{10}$ C) $\frac{6}{7}$ D) $\frac{10}{11}$ E) $\frac{5}{7}$

11. Evaluate $1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{\ddots}}}$.

- A) $\frac{2}{3}$ B) $\frac{3}{2}$ C) 2 D) $\frac{5}{2}$ E) 3

12. Evaluate $\frac{0.\overline{12}}{0.\overline{3}}$.

- A) 27 B) 90 C) $\frac{11}{30}$ D) 4 E) $\frac{30}{11}$

13. Evaluate $0.5 - \frac{0.2}{0.5 + \frac{0.2}{1-0.6}}$.

- A) -0.1 B) -0.2 C) 0 D) 0.2 E) 0.3

14. Given $x > -1$, simplify $|x + 1| + |x + 2| - (2x - 1)$.

- A) 2 B) 3 C) 4 D) 5 E) 6

15. The sum of a group of three consecutive odd numbers is 39. What is the smallest number in the group?

- A) 9 B) 11 C) 13 D) 15 E) 17

16. If $\frac{1}{z} - \frac{1}{z+3} = a + z$, find a in terms of z .

- A) $\frac{z^3 - 3z^2}{z+3}$ B) $\frac{3 - z^3 - 3z^2}{z+3}$
 C) $\frac{3 - z - 3z^2}{z(z+3)}$ D) $\frac{3 - z^3 - 3z^2}{z(z+3)}$
 E) $\frac{z^3 - 3z^2 - 3}{z(z-3)}$

CHAPTER REVIEW TEST 1B

1. $x, y, z \in \mathbb{R}^+$ such that $xy = 24$, $xz = 48$ and $yz = 72$. Find $x + y + z$.
A) 20 B) 22 C) 24 D) 26 E) 28
2. $(a + \frac{1}{a})^2 = 5$, $a \in \mathbb{R}^+$ are given. Find $a^3 + \frac{1}{a^3}$.
A) $2\sqrt{5}$ B) $3\sqrt{5}$ C) $5\sqrt{5}$ D) $6\sqrt{5}$ E) $7\sqrt{5}$
3. How many 9s are there at the end of the number $60! - 50! - 1$?
($n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$)
A) 1 B) 3 C) 9 D) 11 E) 12
4. For $m, n \in \mathbb{N}$, $\frac{1}{n} - \frac{1}{n+3} = \frac{1}{mn}$ is given. Which one of the following is a possible value for n ?
A) 28 B) 32 C) 36 D) 52 E) 67
5. ab is a two-digit number with digits a and b . When the digits are reversed, the number increases by 27. Find $a - b$.
A) 1 B) 2 C) 3 D) 4 E) 5
6. For $a, b, c \in \mathbb{Z}$, $0 < c < a < 4$ and $\frac{a}{c} = b$ are given. How many different possible values of b are there?
A) 2 B) 3 C) 4 D) 5 E) 6
7. For $x < 0$, simplify $|x - \sqrt{(x-1)^2}|$.
A) 1 B) $1 - 2x$ C) $-2x - 1$
D) $1 + 2x$ E) $2x - 1$
8. For $a < b$ and $c < 0$, which one of the following is false?
A) $2a < (a + b)$ B) $(a + b) < 2b$ C) $ac < bc$
D) $(a + c) < (b + c)$ E) $a < \frac{a+b}{2} < b$



9. a and b are digits in two decimal numbers $0.\overline{ab}$ and $0.b$ such that $0.\overline{ab} + 0.b = 0.3\overline{1}$. Find $a + b$.

A) 1 B) 3 C) 9 D) 11 E) 12

10. Simplify $\frac{|x-1|}{|2-x|} - \frac{|x-3|}{|x-2|}$ for $x < 1$.

A) $\frac{2}{2-x}$ B) $\frac{2}{x-2}$ C) $\frac{1}{x-2}$
D) $\frac{1}{2-x}$ E) $\frac{3}{2-x}$

11. Given that n is any natural number greater than 100, which one of the following always divides the number $n^2(n^2 - 1)$?

A) 10 B) 24 C) 30 D) 36 E) 60

12. a and b are integers such that $-2 \leq a < 5$ and $-4 < b \leq 6$. Find the greatest possible value of $3a - 2b$.

A) 15 B) 16 C) 17 D) 18 E) 19

13. $a, b, c \in \mathbb{N}$ such that $ab = 72$ and $bc = 99$. Find the minimum possible value of $a + b + c$.

A) 28 B) 27 C) 26 D) 25 E) 24

14. $a, b, c, d \in \mathbb{Z}^+$ and

$$\begin{array}{r} a \overline{)b} \\ \underline{15} \\ 7 \end{array} \quad \begin{array}{r} b \overline{)6} \\ \underline{c} \\ 3 \end{array} \quad \begin{array}{r} a \overline{)18} \\ \underline{d} \\ x \end{array} \quad \text{are given.}$$

What is the value of x ?

A) 12 B) 13 C) 14 D) 15 E) 16

15. $n \in \mathbb{Z}$ and $n \equiv 2 \pmod{15}$ are given. Find the remainder when n^4 is divided by 15.

A) 0 B) 1 C) 2 D) 5 E) 6

16. Solve $2201^{1801} \equiv x \pmod{7}$ for x .

A) 1 B) 2 C) 3 D) 4 E) 5

CHAPTER REVIEW TEST 1C

- For $a, b \in \mathbb{Z}$, $\frac{b-1}{a} \div (1 - \frac{1}{b}) = x$, $2 < x < 5$ and $a + b = 8$ are given. Find $b - a$.
A) 6 B) 4 C) 0 D) -2 E) -6
- Given $x = \frac{3}{19} + \frac{4}{25} + \frac{5}{31}$ and $y = \frac{1}{57} + \frac{1}{75} + \frac{1}{93}$, find y in terms of x .
A) $1 - 3x$ B) $1 - 2x$ C) $\frac{1-3x}{2}$
D) $\frac{1-2x}{2}$ E) $\frac{2-3x}{2}$
- a is the smallest positive integer such that $a = 4x + 3 = 6y + 5 = 15z + 2$. Find $x + y + z$.
A) 19 B) 20 C) 21 D) 22 E) 23
- When the four-digit number $7a2b$ is divided by 45, the remainder is 13. Find the sum of the possible values of a .
A) 5 B) 6 C) 7 D) 8 E) 9
- Find the remainder when $3^1 + 3^2 + \dots + 3^{1000}$ is divided by 27.
A) 12 B) 17 C) 19 D) 23 E) 26
- Given $x^2 - 2x - 2 = 0$, find $x^2 + \frac{4}{x^2}$.
A) 6 B) 8 C) 10 D) 12 E) 14
- $A = \frac{3}{7} + \frac{4}{11} + \frac{5}{13}$ and $B = \frac{2}{7} + \frac{1}{11} + \frac{2}{13}$ are given. Which one of the following is an integer?
A) $7A + 11B$ B) $11A - 7B$ C) $7B - 2A$
D) $7A - 2B$ E) $3A - B$
- Find the maximum possible value of $\frac{|2x| + 10}{|x| + |x-1|}$.
A) 12 B) 13 C) 14 D) 15 E) 16



9. Find all the possible values of the natural number n which make $\frac{3n+21}{n+3}$ a natural number.

- A) $\{1, 2, 3\}$ B) $\{1, 3, 9\}$ C) $\{3, 9\}$
 D) $\{1, 3, 6\}$ E) $\{0, 1, 3, 9\}$

10. a, b, c are digits in a two-digit number ab and a three-digit number cab such that $(ab)^2 = cab$. Find ab .

- A) 15 B) 16 C) 25 D) 26 E) 31

11. If $3.\bar{3} + 4.\bar{4} + 5.\bar{5} = x - 0.\bar{3}$, find x .

- A) $\frac{31}{3}$ B) $\frac{37}{3}$ C) $\frac{41}{3}$ D) $\frac{46}{3}$ E) $\frac{49}{3}$

12. For which values of x is the expression $\frac{|x - |x||}{x}$ always a positive integer?

- A) $x \in \mathbb{R}^+$ B) $x \in \mathbb{R}^-$ C) $x \in \mathbb{R} - \{0\}$
 D) x is an even integer E) $\emptyset$

13. a and b are digits in the two-digit numbers ab and ba such that $ab - ba = 72$. Find $a^2 + b^2$.

- A) 52 B) 62 C) 72 D) 82 E) 92

14. m and n are two positive integers such that $m + n + mn = 24$. Find $m + n$.

- A) 6 B) 8 C) 10 D) 12 E) 16

15. For $n \in \mathbb{N}$, find the last digit of $1923^{8n+1924}$.

- A) 9 B) 7 C) 3 D) 1 E) 0

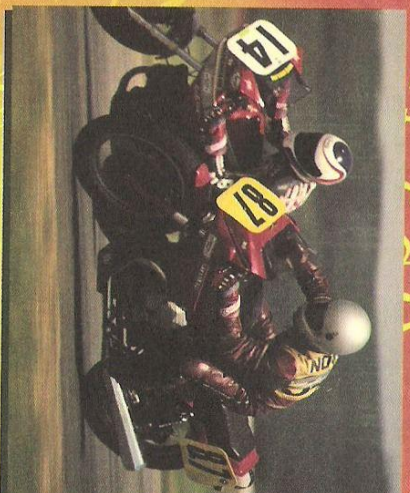
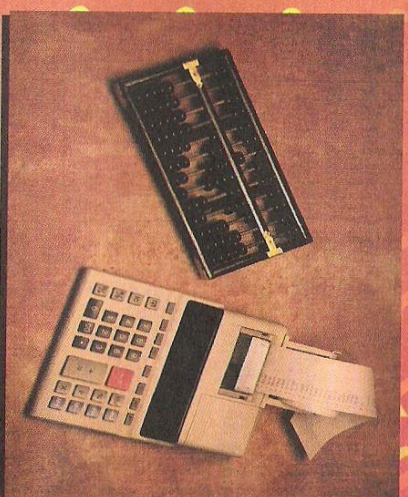
16. For $n \in \mathbb{N}$, find the last digit of 3546^{n+3} .

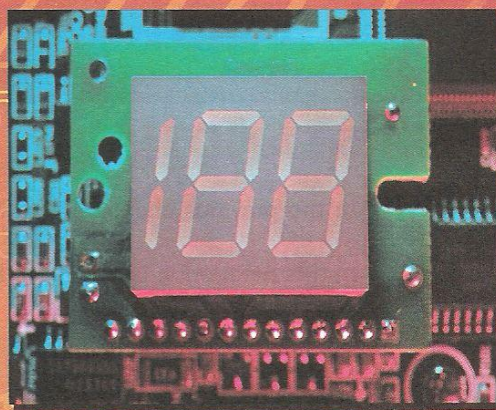
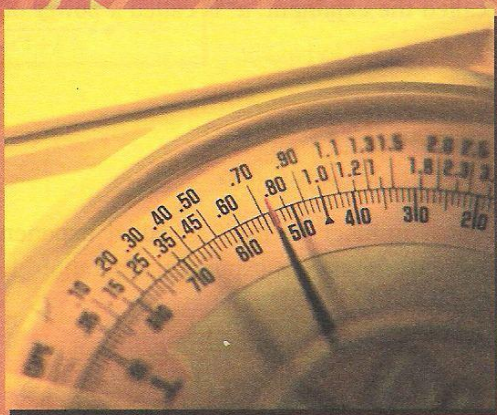
- A) 5 B) 6 C) 7 D) 8 E) 9

200 per 100 per

Expenses Table Example:
 The following table shows the expenses of a business for the month of January. The expenses are listed in dollars and cents. The total expenses for the month are \$1,170.00. The expenses are listed in the following order: 1. Rent, 2. Utilities, 3. Insurance, 4. Advertising, 5. Travel, 6. Entertainment, 7. Office Supplies, 8. Postage, 9. Telephone, 10. Other.

	1 hour	3 hours	5 hours	10 hours
58	\$ 70	\$ 118	\$ 169	\$ 304
56	65	103	142	265
7	73	97	125	180
59	63	125	172	233
3	63	125	180	233
58	62	101	129	229
59	83	93	138	225
67	81	126	199	246
130	66	106	176	
95	79	112	189	
	67	119		





Chapter 12,

EXPONENTS AND RADICAL EXPRESSIONS

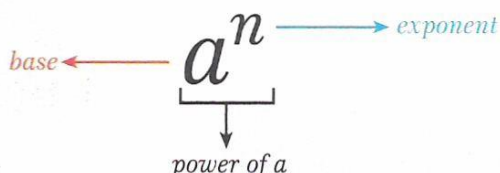
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INTEGER EXPONENTS

A. UNDERSTANDING INTEGER EXPONENTS

In the previous chapter we saw that for any natural numbers a and n , $a^n = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$.

In this notation, a is called the base and n is called the exponent. a^n is called a power of a .



This short-hand way of writing the product of a number by itself is called **exponential notation**. $a \cdot a$ is an expression and a^2 is the same expression in exponential notation. Look at some examples of exponential notation:

$$a^2 = a \cdot a, \quad \text{read as 'a to the second power' or 'a squared'}$$

$$a^3 = a \cdot a \cdot a, \quad \text{read as 'a to the third power' or 'a cubed'}$$

$$a^4 = a \cdot a \cdot a \cdot a, \quad \text{read as 'a to the fourth power'}$$

Notice that the exponents in these examples are all natural numbers. In this section we will extend the concept of exponent to include integer numbers. The following definition extends exponential notation to include zero and negative integer exponents.

Definition

integer exponent

For any real number a and any positive integer n ,

$$1. \quad a^n = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

$$2. \quad a^0 = 1 \quad (0^0 \text{ is undefined})$$

$$3. \quad a^{-n} = \frac{1}{a^n} \quad \text{where } a \neq 0.$$

For example,

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

$$0.1^{-2} = \frac{1}{0.1^2} = \frac{1}{0.1 \cdot 0.1} = \frac{1}{0.01} = 100$$

$$0.43^1 = 0.43$$

$$(-2)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

$$2006^0 = 1$$

$$5^{-1} = \frac{1}{5}$$

$$(-6)^3 = (-6) \cdot (-6) \cdot (-6) = -216$$

$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$$

Don't confuse
 $-(5^2)$ with $(-5)^2$!
 $-(5^2)$ is the opposite of $(-5)^2$.
 $-(5^2) = -25$,
 $(-5)^2 = (-5) \cdot (-5) = 25$.



EXAMPLE 1

Remove the negative exponent(s) in each expression.

a. $(x^3y^5)^0$

b. y^{-5}

c. $\frac{a^{-2}}{b^{-5}}$

d. $\frac{x^4(y+1)^{-2}}{2^{-2}z^3}$

e. $\frac{1}{a^{-2}+b^{-2}}$

Solution

a. $(x^3y^5)^0 = 1$

b. $y^{-5} = \frac{1}{y^5}$

c. $\frac{a^{-2}}{b^{-5}} = \frac{\frac{1}{a^2}}{\frac{1}{b^5}} = \frac{1}{a^2} \cdot \frac{b^5}{1} = \frac{b^5}{a^2}$

d. $\frac{x^4(y+1)^{-2}}{2^{-2}z^3} = \frac{x^4 \cdot \frac{1}{(y+1)^2}}{\frac{1}{2^2} \cdot z^3} = \frac{4x^4}{(y+1)^2 z^3}$

e. $\frac{1}{a^{-2}+b^{-2}} = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{1}{\frac{a^2+b^2}{a^2b^2}} = \frac{a^2b^2}{a^2+b^2}$

Be careful!

$$\frac{1}{a^{-2}+b^{-2}} \neq a^2+b^2$$

Note

When we move expressions from the numerator to the denominator (or from the denominator to the numerator), we simply change the sign of the exponent.

EXAMPLE 2

Evaluate the expressions.

a. $(-2)^3 + (-2^4) + (-2)^4 + 2^0 - 2^{-2}$

b. $\left(\frac{3}{2}\right)^{-1} + \frac{2^{-1}}{3^{-1}}$

Solution a. $(-2)^3 + (-2^4) + (-2)^4 + 2^0 - 2^{-2} = (-8) + (-16) + 16 + 1 - \frac{1}{2^2} = -7.25$

b. $\left(\frac{3}{2}\right)^{-1} + \frac{2^{-1}}{3^{-1}} = \frac{1}{3} + \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{1}{3} + \frac{3}{2} = \frac{13}{6}$

Note

1. All powers of a positive real number are positive. All even powers of a negative real number are positive, and all odd powers of a negative real number are negative.
2. $(-a)^n$ and $-(a^n)$ are not the same. In $(-a)^n$ the base is $-a$, but in $-(a^n)$ the base is a .

Check Yourself 1

1. Evaluate $0.5^{-3} + 1^{2006} - (-2)^4$.
2. Remove the negative exponent(s) in each expression.

a. $\frac{3^{-3}c^2}{2^{-1}a^{-2}b^{-3}}$

b. $\frac{m^2n^2p^{-4}}{81^{-1}b^2}$

c. $\frac{1}{(x+y)^{-2}}$

Answers

1. -7
2. a. $\frac{2a^2b^3c^2}{27}$ b. $\frac{81m^2n^2}{b^2p^4}$ c. $(x+y)^2$



B. WORKING WITH INTEGER EXPONENTS

1. The Product Rule

Consider the product $a^m \cdot a^n$. We have already seen that for positive integers m and n ,

$$a^m \cdot a^n = (\underbrace{a \cdot a \cdot a \cdots a}_{m \text{ times}}) \cdot (\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}) = (\underbrace{a \cdot a \cdot a \cdots a}_{m+n \text{ times}}) = a^{m+n}.$$

We can prove that this is also true for negative integers by using $a^m = \frac{1}{a^{-m}}$ and $a^n = \frac{1}{a^{-n}}$. We can see that to multiply exponential expressions with the same base, we keep the same base and add the exponents. For example, $a^7 \cdot a^{-2} = a^{7+(-2)} = a^5$.

PRODUCT RULE FOR EXPONENTS

$$a^m \cdot a^n = a^{m+n} \quad (a \in \mathbb{R} \text{ and } m, n \in \mathbb{Z})$$

EXAMPLE

3

Simplify each expression, removing any negative exponents as needed.

a. x^6x^{-3}

b. $16b^{-5}a^3a^{-1}b^5$

c. $(y+1)^3(y+1)(y+1)^{-7}$

d. $3x^{2m-1}x^{3m+2}$

Solution

a. $x^6x^{-3} = x^{6+(-3)} = x^3$

b. $16b^{-5}a^3a^{-1}b^5 = 16a^{3+(-1)}b^{(-5)+5} = 16a^2b^0 = 16a^2$

c. $(y+1)^3(y+1)(y+1)^{-7} = (y+1)^{3+1+(-7)} = (y+1)^{-3} = \frac{1}{(y+1)^3}$

d. $3x^{2m-1}x^{3m+2} = 3x^{2m-1+3m+2} = 3x^{5m+1}$

Note

We can only apply the product rule when we are multiplying expressions with the same base. We cannot use the product rule to simplify the product of two powers which have different bases.

2. The Quotient Rule

Consider the quotient $\frac{a^m}{a^n}$, where m and n are integers.

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n} = a^{m-n}$$

We can see that to divide two exponential expressions with the same base, we keep the same base and subtract the exponent in the denominator from the exponent in the numerator. For example, $\frac{a^4}{a^5} = a^{4-5} = a^{-1} = \frac{1}{a}$.

QUOTIENT RULE FOR EXPONENTS

$$\frac{a^m}{a^n} = a^{m-n} \quad (a \in \mathbb{R} \setminus \{0\} \text{ and } m, n \in \mathbb{Z})$$



EXAMPLE**4** Simplify each expression, removing any negative exponents as needed.

a. $\frac{x^{-2}}{x^{-5}}$

b. $\frac{a^{3m-1} \cdot a^2}{a^{2m+4}}$

c. $\frac{a^5 bc^6}{b^3 c}$

d. $\frac{12(t+1)^{12} a^3}{a(t+1)^7}$

Solution

a. $\frac{x^{-2}}{x^{-5}} = x^{-2-(-5)} = x^3$

b. $\frac{a^{3m-1} \cdot a^2}{a^{2m+4}} = a^{(3m-1)+2-(2m+4)} = a^{m-3}$

c. $\frac{a^5 bc^6}{b^3 c} = a^5 b^{1-3} c^{6-1} = a^5 b^{-2} c^5 = \frac{a^5 c^5}{b^2}$

d. $\frac{12(t+1)^{12} a^3}{a(t+1)^7} = 12a^{3-1}(t+1)^{12-7} = 12a^2(t+1)^5$

Note

We can apply the quotient rule when we are dividing expressions with the same base. We cannot use the quotient rule to simplify the quotient of two powers which have different bases.

3. Power Rules

Consider the exponential expression $(a^m)^n$ where m and n are positive integers.

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot a^m \cdots a^m}_{n \text{ times}} = a^{\overbrace{m+m+\cdots+m}^{n \text{ times}}} = a^{m \cdot n}$$

We can see that to raise an exponential expression to a power, we keep the same base and multiply the exponents. For example, $(a^2)^3 = a^{(-2) \cdot 3} = a^{-6}$.

Next, consider the exponential expression $(a \cdot b)^m$ where m is a positive integer.

$$(a \cdot b)^m = \underbrace{(a \cdot b) \cdot (a \cdot b) \cdots (a \cdot b)}_{m \text{ times}} = \underbrace{(a \cdot a \cdot a \cdots a)}_{m \text{ times}} \cdot \underbrace{(b \cdot b \cdot b \cdots b)}_{m \text{ times}} = a^m \cdot b^m$$

We can see that to raise a product to a power, we raise each factor to that power. For example, $(a \cdot b)^3 = a^3 \cdot b^3$.

Finally, let us look at the exponential expression $\left(\frac{a}{b}\right)^m$ where m is a positive integer.

$$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b}}_{m \text{ times}} = \frac{\overbrace{a \cdot a \cdot a \cdots a}^{m \text{ times}}}{\underbrace{b \cdot b \cdot b \cdots b}_{m \text{ times}}} = \frac{a^m}{b^m}$$

So to raise a quotient to a power, we raise both the numerator and denominator to that power.

For example, $\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}$.

Note that we can prove that these three rules are also true for negative integers by using

$$a^m = \frac{1}{a^{-m}} \text{ and } a^n = \frac{1}{a^{-n}}.$$

POWER RULES FOR EXPONENTS

$$1. (a^m)^n = a^{m \cdot n} \quad 2. (a \cdot b)^m = a^m \cdot b^m \quad 3. \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (a, b \in \mathbb{R}, b \neq 0 \text{ and } m, n \in \mathbb{Z})$$

Note

As a consequence of the power rules for exponents, the following properties hold.

$$1. (a^m \cdot b^n)^p = a^{mp} \cdot b^{np} \quad 2. \left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}} \quad 3. \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

EXAMPLE

5 Simplify each expression, removing any negative exponents as needed.

$$\text{a. } (y^{-2})^{-4} \quad \text{b. } (a^2b^{-7})^3 \quad \text{c. } \left(\frac{x^{-2}}{y^3}\right)^2 \quad \text{d. } \left[\frac{(s-2)^2a^3}{2b^{-4}}\right]^{-5}$$

Solution a. $(y^{-2})^{-4} = y^{(-2)(-4)} = y^8$

b. $(a^2b^{-7})^3 = a^{2 \cdot 3}b^{(-7) \cdot 3} = a^6b^{-21} = \frac{a^6}{b^{21}}$

c. $\left(\frac{x^{-2}}{y^3}\right)^2 = \frac{x^{(-2) \cdot 2}}{y^{3 \cdot 2}} = \frac{x^{-4}}{y^6} = \frac{1}{x^4y^6}$

d. $\left[\frac{(s-2)^2a^3}{2b^{-4}}\right]^{-5} = \frac{(s-2)^{2(-5)} \cdot a^{3(-5)}}{2^{-5} \cdot b^{(-4)(-5)}} = \frac{(s-2)^{-10}a^{-15}}{2^{-5}b^{20}} = \frac{32}{(s-2)^{10}a^{15}b^{20}}$

Common Mistakes

$$x^2 \cdot x^3 \neq x^{2 \cdot 3}$$

$$(x^2)^5 \neq x^{2+5}$$

$$\frac{x^6}{x^3} \neq x^{6 \div 3}$$

$$(3x)^2 \neq 3x^2$$

$$(x^2 + y^3)^4 \neq x^{2 \cdot 4} + y^{3 \cdot 4} \quad \frac{1}{x^{-2} + y^{-3}} \neq x^2 + y^3$$

Check Yourself 2

Simplify each expression, removing any negative exponents as needed.

1. $(2x^5y^{-2})^{-3}$
2. $(\frac{a^2}{b^5})^{-5}$
3. $\frac{15m^{-3}n^{-2}}{6m^{-4}n^5}$
4. $(\frac{a^{-2}}{b^{-5}b^5})^{-5}$
5. $[\frac{(a+3)^2c^2b^{-5}}{(a+3)^{-3}b}]^{-1}$
6. $\frac{(2^3a^{-2}b^3)^{-2}}{9^{-1}ac^3}$

Answers

1. $\frac{y^6}{8x^{15}}$
2. $\frac{b^{25}}{a^{10}}$
3. $\frac{5m}{2n^7}$
4. a^{10}
5. $\frac{b^6}{(a+3)^5c^2}$
6. $\frac{9a^3}{64b^6c^3}$

EXAMPLE 6

Simplify each expression, removing any negative exponents as needed.

- a. $(\frac{x^4}{x^3x^2})^{-2} \cdot \frac{x^3 \div x^2}{x}$
- b. $(\frac{1}{2}a^{-1}b^3)^{-3} \div (a^{-2}b^{-8})$
- c. $(3a^{-2}b^{-1}c^5b^{-2})^2 \cdot (\frac{a^{-2}b^3c^4}{4b^6})^{-2}$

Solution

$$a. (\frac{x^4}{x^3x^2})^{-2} \cdot \frac{x^3 \div x^2}{x} = (\frac{x^4}{x^5})^{-2} \cdot \frac{x}{x} = (x^{-1})^{-2} \cdot 1 = x^2$$

$$b. (\frac{1}{2}a^{-1}b^3)^{-3} \div (a^{-2}b^{-8}) = (\frac{1}{2})^{-3}a^3b^{-9} \div (a^{-2}b^{-8}) = \frac{8a^3b^{-9}}{a^{-2}b^{-8}} = 8a^5b^{-1} = \frac{8a^5}{b}$$

$$c. (3a^{-2}b^{-1}c^5b^{-2})^2 \cdot (\frac{a^{-2}b^3c^4}{4b^6})^{-2} = (3a^{-2}b^{-3}c^5)^2 \cdot (\frac{a^{-2}b^{-3}c^4}{4})^{-2} = 9a^{-4}b^{-6}c^{10} \cdot \frac{a^4b^6c^{-8}}{4^{-2}} = 144c^2$$

EXAMPLE 7

Simplify each expression, removing any negative exponents as needed.

- a. $\frac{a^{-1} + b^{-1}}{a^{-2} - b^{-2}}$
- b. $a^6(a^{-2} - a^{-4})(a^2 + a^3)^{-1}$

Solution

$$a. \frac{a^{-1} + b^{-1}}{a^{-2} - b^{-2}} = \frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}} = \frac{\frac{b+a}{ab}}{\frac{b^2 - a^2}{a^2b^2}} = \frac{b+a}{ab} \cdot \frac{a^2b^2}{(b-a)(b+a)} = \frac{ab}{b-a}$$

$$b. a^6(a^{-2} - a^{-4})(a^2 + a^3)^{-1} = a^6(\frac{1}{a^2} - \frac{1}{a^4}) \frac{1}{a^2 + a^3} = a^6 \cdot \frac{a^2 - 1}{a^4} \cdot \frac{1}{a^2 + a^3} = \frac{a^6(a-1)(a+1)}{a^4 \cdot a^2(1+a)} = a^{-1}$$



EXAMPLE**8** Evaluate the expressions.

a. $(\frac{1}{2})^2 \cdot 2^{-4} \cdot (\frac{1}{4})^{-3}$

b. $[9^2 \cdot \frac{1}{3^4} \cdot (\frac{1}{9})^{-2} \cdot 3^{-1}] \div 27$

c. $(-\frac{2}{3})^{-2} - (-1.5)^{-2} \cdot (-1.5)^3 - (0.25^{-3})^0 + 81 \cdot (-\frac{2}{3})^4$

Solution We could evaluate each expression term by term. However, it is sometimes quicker and easier to write each term in the same base and then apply the rules for exponential calculations.a. If we write 4 as 2^2 then each term will be a power of 2:

$$(\frac{1}{2})^2 \cdot 2^{-4} \cdot (\frac{1}{4})^{-3} = (2^{-1})^2 \cdot 2^{-4} \cdot (2^{-2})^{-3} = 2^{-2} \cdot 2^{-4} \cdot 2^6 = 2^{-2-4+6} = 2^0 = 1.$$

b. This time we write each number as a power of 3.

$$[9^2 \cdot \frac{1}{3^4} \cdot (\frac{1}{9})^{-2} \cdot 3^{-1}] \div 27 = [(3^2)^2 \cdot 3^{-4} \cdot (3^{-2})^{-2} \cdot 3^{-1}] \div 3^3 \\ = [3^4 \cdot 3^{-4} \cdot 3^4 \cdot 3^{-1}] \div 3^3 = 3^{4-4+4-1-3} = 3^0 = 1.$$

c. This time we cannot write all the terms in the same base. Instead, let us evaluate each term separately.

$$(-\frac{2}{3})^{-2} = (-\frac{3}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4} = 2.25$$

$$(-1.5)^{-2} \cdot (-1.5)^3 = (-1.5)^1 = -1.5$$

$$(0.25^{-3})^0 = 1$$

$$81 \cdot (-\frac{2}{3})^4 = 81 \cdot (\frac{2}{3})^4 = 3^4 \cdot \frac{2^4}{3^4} = 2^4 = 16$$

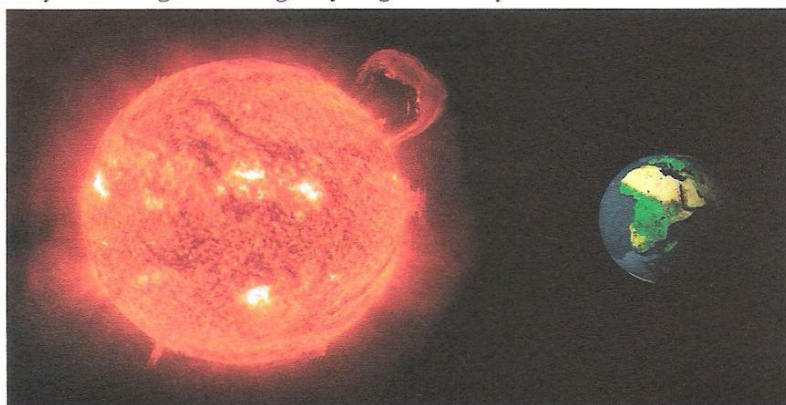
$$\text{So } (-\frac{2}{3})^{-2} - (-1.5)^{-2} \cdot (-1.5)^3 - (0.25^{-3})^0 + 81 \cdot (-\frac{2}{3})^4 = 2.25 - (-1.5) - 1 + 16 = 18.75.$$

Check Yourself 31. Simplify $(\frac{a^2b}{2^3b^{-1}c})^{-2} \div (-2a^2ba^{-3})^4$, removing any negative exponents as needed.2. Simplify $(x^{-1} + y^{-1})(\frac{x+y}{xy})^{-2}$, removing any negative exponents as needed.3. Evaluate $\frac{3^{-3} \cdot 9^{-3}}{27^{-2}}$.**Answers**

1. $\frac{4c^2}{b^8}$ 2. $\frac{xy}{x+y}$ 3. $\frac{1}{27}$



C. SCIENTIFIC NOTATION

[illegible]

scientific notation, standard notation

A number is expressed in **scientific notation** when it is in the form $a \cdot 10^n$, where $1 < |a| < 10$ and n is an integer.

If a number is written in normal notation, we say it is in **standard notation**.

For example, 500 000 is a number in standard notation. $5 \cdot 10^5$ is the same number in scientific notation.

To write a number in scientific notation, follow the steps.

1. Place a decimal point so that the number is between 1 and 10.
2. Count how many decimal places the decimal point is from its old position (for example, x).
3. Multiply by 10^x .
 - a. If the number became smaller in step 1, multiply by a positive power.
 - b. If the number became larger in step 1, multiply by a negative power.

For example, as we have seen, the distance between the Earth and the Sun is 149 600 000 km. To express this number in scientific notation we proceed as follows:

$$149600000. = 1.49600000 \cdot 10^8 = 1.496 \cdot 10^8.$$

8 decimal places

Place a decimal point so that the number is between 1 and 10.

Count how many decimal places your decimal point is from its old position (= eight).

Multiply by the corresponding power of 10.

EXAMPLE**9**

Write each number in scientific notation.

a. 0.00000542

b. 6023000000000000

c. 0.000000000000007

Solutiona. First we place a decimal point so that the number is between 1 and 10: **5.42**.Next we count how many decimal places our decimal point is from its old position: **6**.Finally, we multiply 5.42 by 10^{-6} since the number increased in the first step.

So $0.00000542 = 5.42 \cdot 10^{-6}$.

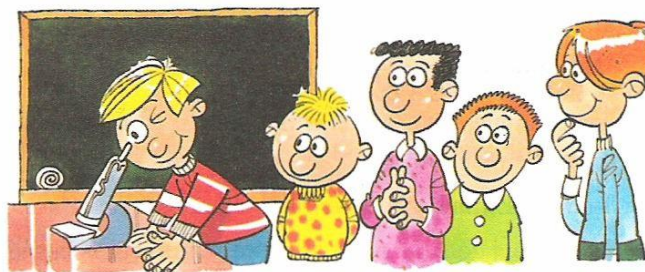
b. By the same process, $6023000000000000 = 6.023 \cdot 10^{15}$.

c. $0.000000000000007 = 7 \cdot 10^{-13}$

EXAMPLE**10**Use scientific notation to calculate $\frac{420000000 \cdot 0.0002}{14000000}$.**Solution**

First we write each number in scientific notation and then we work with the numbers and exponential expressions separately.

$$\frac{420000000 \cdot 0.0002}{14000000} = \frac{4.2 \cdot 10^8 \cdot 2 \cdot 10^{-4}}{1.4 \cdot 10^7} = \frac{4.2 \cdot 2}{1.4} \cdot 10^{8-4-7} = 6 \cdot 10^{-3} = 0.006$$



Wow! What a scientific number!

Check Yourself 4

1. Write each number in scientific notation.

a. 890000000

b. 0.00045

2. Use scientific notation to calculate $\frac{0.0000036 \cdot 400000000}{0.0012}$.**Answers**

2. $1.2 \cdot 10^6$



EXERCISES 2.1

A. Understanding Integer Exponents

1. Evaluate the expressions.

- | | |
|-------------------------|---------------------------------|
| a. 4^3 | b. 3^4 |
| c. -3^2 | d. $(-3)^2$ |
| e. $(-3)^{-2}$ | f. $(3 \cdot 2 + 5 \cdot 75)^0$ |
| g. $(\frac{1}{2})^{-2}$ | h. $(\frac{1}{2})^{-3}$ |
| i. $(-2)^{-2}$ | j. -7^0 |
| k. $(-7)^0$ | l. $(-\frac{25}{4})^0$ |
| m. 2^{-2} | n. $(-2)^{-7}$ |
| o. $(0.25)^{-1}$ | p. 0.01^{-2} |
| q. 0.1^2 | r. -0.1^{-2} |
| s. $(1.75^{-3})^0$ | t. $(1.75^0)^{-3}$ |
| u. $(-1)^{1652}$ | v. -1^{1652} |

B. Working with Integer Exponents

2. Simplify the expressions, removing any negative exponents as needed.

- | | |
|---------------------------------|------------------------------------|
| a. $(x^2y^3)^0$ | b. $x^{-4}y^4$ |
| c. $x^{-3}y^{-2}$ | d. $\frac{x^{-1}}{x-y}$ |
| e. $\frac{x^{-3}}{y^{-3}}$ | f. $\frac{(x+y)^{-2}}{(x+y)^{-1}}$ |
| g. $[(\frac{x-y}{x+y})^0]^{-2}$ | h. $\frac{x^{-2}y^{-3}}{y^2x^2}$ |
| i. $\frac{1}{(x-y)^{-6}}$ | j. $(\frac{x+y}{x-y})^{-2}$ |

3. Evaluate $\frac{(-2)^{-2004} - (-2)^{-2005}}{-2^{-2006}}$.

4. Determine whether each statement is true or false.

- | |
|--|
| a. $3^{-1} \cdot 4^{-1} = (3 \cdot 4)^{-1}$ |
| b. $2^{-1} + 3^{-1} = (2 + 3)^{-1}$ |
| c. $(3^{-1} \cdot 4^{-1})^2 = 3^{-2} \cdot 4^{-2}$ |
| d. $(2^{-1} + 3^{-1})^2 = 2^{-2} + 3^{-2}$ |
| e. $2^{-2} \cdot 3^{-2} = (2 \cdot 3)^{-2}$ |
| f. $3^{-2} + 4^{-2} = (3 + 4)^{-2}$ |
| g. $3^{-2} \cdot 4^{-2} = \frac{1}{3^2 \cdot 4^2}$ |
| h. $3^{-2} + 4^{-2} = \frac{1}{3^2 + 4^2}$ |

5. Simplify the expressions, removing any negative exponents as needed.

- | | |
|---|--|
| a. $\frac{-2x^3y^2}{8x^5y^5}$ | b. $\frac{50x^3y^{-4}x^{-1}}{30x^{-2}y^3}$ |
| c. $\frac{8x^{-10}x^{10}}{4y^{-12}y^4}$ | d. $\frac{x^8x^{-2}}{y^{-4}y^6}$ |
| e. $2x^4 \cdot (-2x)^4$ | f. $\frac{25x^{-3}y^{-2}}{5x^2y}$ |
| g. $(3x^2y^{-3})^{-1}$ | h. $(\frac{a^2}{b^{-3}})^2$ |
| i. $\frac{30m^{-6}n^{-4}}{12m^{-5}n^{10}m^{-3}}$ | j. $(\frac{a^{-4}}{b^{-10}b^{10}})^{-10}$ |
| k. $(\frac{x^2}{y^4})^{-5} \cdot x^6$ | l. $(x-y)^2(x-y)^{-1}(x-y)^3$ |
| m. $\frac{x^{-4}(y-x)^4}{y^{-2}(x-y)^4}$ | n. $\frac{(x-1)^4(1-x)^2}{-(x-1)^{-2}(1-x)}$ |
| o. $\frac{32x^{-8}x^{11}}{24y^{-12}y^{12}}$ | p. $\frac{(a-b)^{-1}(a+b)^{-1}}{(a^2-b^2)^{-2}}$ |
| q. $[(\frac{x^3y^{-1}z^{-2}}{x^{-1}y^{-1}z})^{-1}]^2$ | r. $[(\frac{x^{-2}y^4z}{x^{-3}y^{-2}z^2})^2]^{-1} \cdot (x^{-2}y)^3$ |

6. Simplify the expressions, removing any negative exponents as needed.

a. $\frac{1-a^{-1}}{1-a^{-2}}$

b. $\frac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$

c. $\frac{1-x}{-1+x^{-1}}$

d. $[\frac{x^{-2}-y^{-2}}{(x^{-1}+y^{-1})^2}]^{-1}$

e. $\frac{x-y}{x^{-1}-y^{-1}}$

f. $\frac{x^{-3}-y^{-3}}{x^{-1}-y^{-1}}$

g. $\frac{xy(x^{-1}-y^{-1})}{x^2-y^2}$

h. $\frac{x^{-2}-y^{-2}}{x+y}$

7. Write each expression in its simplest form.

a. $x^2(x^{n-3} + x^{n-2}) + x^n(x^{-1} - 1)$

b. $x^n(x^{m-n+1} + x^{m-n+1} + x^{m-n-1})$

c. $(x^m)^n \cdot (x^n)^{n-m}$

d. $x^{2n} \cdot x^{n+2} \cdot x^{2n-1}$

e. $x(x^{n+1} + x^n + x^{n-1}) - x^n(x^2 + 1)$

f. $x^{m-n}(x^{m+n} + x^n)$

g. $(x^{m-n})^m(x^{m+n})^n$

8. Simplify each expression, removing any negative exponents as needed.

a. $(-x^{-2})^{-1} \cdot (\frac{1}{x})^3 \cdot (-x^{-3}) \cdot (\frac{1}{x})^{-2}$

b. $(-x^{-1})^{-1} \cdot (-x)^{-1} \cdot (-x^{-2})^{-2} \cdot (\frac{1}{x^{-2}})^{-3} \cdot x^{-4}$

c. $\frac{(-x^{-m})^{-2} \cdot (x^m)^{-1} \cdot (x^{m+2})^{-1}}{-x^{-2} \cdot x^{-m+1} \cdot x^{m-1}}$

9. Simplify $\frac{7^{n+1} - 3 \cdot 7^{n-1}}{7^n - 3 \cdot 7^{n-2}}$.

10. Simplify $\frac{-x^{4n+3} \cdot (-x^{4n-3})}{(-x)^{3-5n} \cdot (-x)^{12n}}$ if n is even.

C. Scientific Notation

11. Evaluate $\frac{0.0001^{-6} \cdot 0.0002^7}{0.8^3 \cdot 0.0002^{-1}}$.

12. Evaluate $\frac{100^{20} + 100^{25} + 1000^{15}}{10^{33} + 10^{43} + 10^{48}}$.

13. The distance from the Earth to the Sun is about 150 000 000 kilometers. How long would it take a rocket traveling at $3 \cdot 10^3$ kilometers per hour to reach the Sun?



Mixed Problems

14. Given $x = 0.5^3$ and $y = 0.25^4$, evaluate

$(x + \frac{1}{y})^2 - (x - \frac{1}{y})^2$.

15. Given $5^{x+1} = 7^{x+2}$, evaluate $(\frac{7}{5})^{x+3}$.

16. Given $2^{-a} = x$ and $3^a = y$, express 162^{a+1} in terms of x and y .

17. Simplify $\frac{2}{1+2^{a-b-1}} + \frac{2}{1+2^{b-a+1}}$.

18. Given $2^x + 2^{-x} = 3$, evaluate $8^x + 8^{-x}$.

19. Simplify $\frac{1}{25^x + 5^x + 1} + \frac{1}{5^x + 1 + 5^{-x}} + \frac{1}{1 + 5^{-x} + 25^{-x}}$.

20. How many digits does $125^{10} \cdot 32^8 \cdot 150^5$ have?

21. Order the numbers.

$x = 2^{-45}, y = 3^{-36}, z = 5^{-27}, t = 6^{-18}$.



2

SQUARE ROOTS

A. UNDERSTANDING SQUARE ROOTS

Recall that a^2 is the square of a . The operation of finding the square of a number is called **squaring** the number. The reverse of this operation is called **taking the square root** of the number. Taking the square root of a number x means finding the number whose square is x .

Look at some examples of square roots:

4 is a square root of 16 since $4^2 = 16$.

-4 is also a square root of 16 since $(-4)^2 = 16$.

0 is a square root of 0 since $0^2 = 0$.

$3a$ is a square root of $9a^2$ since $(3a)^2 = 9a^2$.

$-3a$ is also a square root of $9a^2$ since $(-3a)^2 = 9a^2$.

We can see that all positive numbers except zero have two square roots, one positive and the other negative. The non-negative square root of a number is called the **principal square root** of that number. For example, -5 and 5 are square roots of 25, but only 5 is the principal square root of 25. In the rest of this chapter, we will concentrate on the principal square root of a number.

Definition

square root, principal square root

Let a be a non-negative real number. If $b^2 = a$ then b is a **square root** of a . The non-negative square root of a is called the **principal square root** of a .

Notation

The symbol $\sqrt{\quad}$, called a **radical sign**, is used to represent the square root of a number. The number under the radical sign is called a **radicand**. An expression which comprises a radical sign and a radicand is called a **radical expression**.



Be careful
 $-\sqrt{36} \neq \sqrt{-36}$

The radical sign generally indicates the principal square root of a number: $\sqrt{36}$ means the principal square root of 36 ($=6$). If we want to write the negative square root, we write $-\sqrt{36}$, which is -6 .

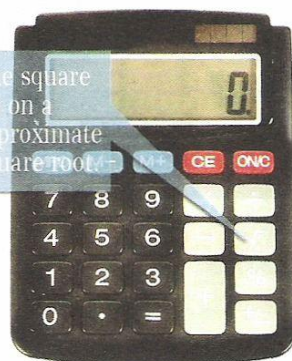
Note that we cannot find a real square root of a negative number because the square of a real number is always positive. Expressions such as $\sqrt{-4}$, $\sqrt{-1}$ and $\sqrt{-200}$ are undefined in the set of real numbers.

Numbers whose square roots are integers or quotients of integers are called **perfect squares**. For example, 100, 25, 16 and zero are perfect squares.

The square roots of numbers that are not perfect squares are irrational numbers.

For example, the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are all irrational numbers.

You can use the square root button on a calculator to approximate an irrational square root.



EXAMPLE 11

Evaluate the following square roots exactly if possible. Otherwise, approximate the square roots and round your answer to two decimal places.

a. $-\sqrt{144}$

b. $\sqrt{2.89}$

c. $\sqrt{0.04}$

d. $\sqrt{7}$

Solution a. $-\sqrt{144} = -12$

b. $\sqrt{2.89} = 1.7$

c. $\sqrt{0.04} = 0.2$

d. $\sqrt{7} = 2.645751311... \approx 2.65$

EXAMPLE 12

Find the solution set of the equation $x^2 = 16$.

Solution We are looking for all numbers whose square is 16. In other words, we are looking for the square roots of 16: $x = \sqrt{16}$ or $x = -\sqrt{16}$. So the solution set is $\{4, -4\}$.

When we are finding the principal square root of an expression with a variable, we must remember that result must be non-negative. Sometimes this is not so obvious. For example, it is not always true that $\sqrt{x^2} = x$, since x may be negative. If x is negative then $\sqrt{x^2} = x$ implies that the square root is negative, which is wrong; in fact, if x is negative then $\sqrt{x^2} = -x$. By the following definition we guarantee that the principal square root is always non-negative as it should be.

Definition

For any real number x , $\sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.

For example, $\sqrt{(-4)^2} = |-4| = 4$, $\sqrt{3^2} = |3| = 3$, $\sqrt{(x-2)^2} = |x-2|$, $\sqrt{a^4} = |a^2| = a^2$.



EXAMPLE 13 Simplify the expression $\sqrt{x^2} + \sqrt{y^2} + 2x$ if $x > 0$ and $y < 0$.

Solution $x > 0$ means $|x| = x$ and $y < 0$ means $|y| = -y$.

$$\text{So } \sqrt{x^2} + \sqrt{y^2} + 2x = |x| + |y| + 2x = x + (-y) + 2x = 3x - y.$$

EXAMPLE 14 Simplify the expression $\sqrt{(x+4)^2} + \sqrt{x^2}$ if $-4 < x < 0$.

Solution $\sqrt{(x+4)^2} + \sqrt{x^2} = |x+4| + |x|$

Note that $x+4 > 0$ and $x < 0$.

$$\text{So we have } |x+4| + |x| = x+4 - x = 4.$$

EXAMPLE 15 Simplify the expression $\sqrt{a^2} + \sqrt{b^2} + \sqrt{b^2 - 2b + 1} + \sqrt{1 - 2x + x^2}$ if $a < 0 < b < 1 < x$.

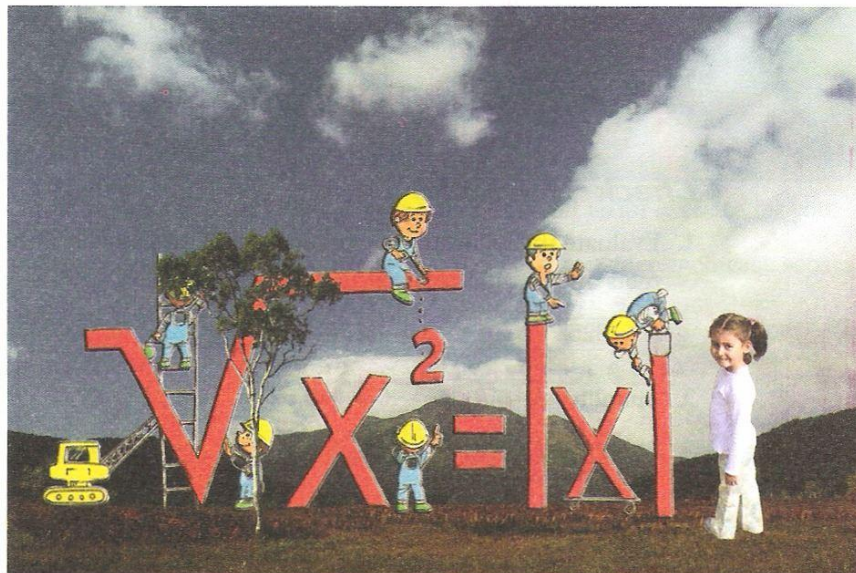
Solution Let us consider each term separately.

$a < 0$ means $\sqrt{a^2} = -a$ and $b > 0$ means $\sqrt{b^2} = b$.

$$\sqrt{b^2 - 2b + 1} = \sqrt{(b-1)^2} = |b-1| = -b+1 = 1-b \text{ since } b < 1, \text{ i.e. } b-1 < 0.$$

$$\sqrt{1 - 2x + x^2} = \sqrt{(1-x)^2} = |1-x| = x-1 \text{ since } x > 1, \text{ i.e. } 1-x < 0.$$

Consequently, the expression will be equal to $-a + b + (1-b) + (x-1) = x - a$.



EXAMPLE 16 Find the real numbers x and y which satisfy the equation $\sqrt{y^2} + \sqrt{x^2 - 2x + 1} = 0$.

Solution $\sqrt{y^2} + \sqrt{x^2 - 2x + 1} = 0$ is equivalent to $\sqrt{y^2} + \sqrt{(x-1)^2} = 0$ i.e. $|y| + |x-1| = 0$.
But $|y| + |x-1| = 0$ only when $|y| = 0$ and $|x-1| = 0$.
So $y = 0$ and $x = 1$.

EXAMPLE 17 Find the real numbers x which satisfies the equation $\sqrt{5-x-x^2} = \sqrt{x^2-9x+5}$.

Solution Notice that neither radicand is a perfect square. Therefore, we cannot simplify the square roots as in the previous examples. Instead, by squaring both sides of the equation, we obtain

$$\begin{aligned}(\sqrt{5-x-x^2})^2 &= (\sqrt{x^2-9x+5})^2 \\5-x-x^2 &= x^2-9x+5 \\2x^2-8x &= 0 \\2x(x-4) &= 0.\end{aligned}$$

So $x = 0$ or $x = 4$.

If we substitute 4 for x into the original equation we get $\sqrt{-15} = \sqrt{-15}$, which seems to be correct at first glance. However, the square root of a negative number is undefined. For this reason, we cannot include 4 in the solution set.

If we substitute zero for x in the original equation, the radicand will be positive. So this is a possible value of x , and the solution set is $\{0\}$.

Note

For any real number x , $\sqrt{x^2} = |x|$ but $(\sqrt{x})^2 = x$.

Check Yourself 5

1. Evaluate the following square roots exactly if possible. Otherwise, approximate the square roots and round the result to two decimal places.

a. $\sqrt{625}$ b. $-\sqrt{5}$ c. $\sqrt{1.21}$ d. $\sqrt{0.81}$

2. Simplify $\sqrt{1-4b+4b^2}$ if $b > 0.5$.

3. Simplify $\sqrt{x^2} + \sqrt{y^2} - \sqrt{x^2 - 2xy + y^2}$ if $x < y < 0$.

Answers

1. a. 25 b. -2.24 c. 1.1 d. 0.9 2. $2b-1$ 3. $-2y$

B. PROPERTIES OF SQUARE ROOTS

In the age of computers, calculating or approximating the square root of a number is very easy. In this section we will concentrate on the algebraic properties of square roots, which are useful for solving equations which contain square roots.

PROPERTIES OF SQUARE ROOTS

For any $a \geq 0$, $b \geq 0$ and $n \in \mathbb{Z}$,

1. $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ (e.g. $\sqrt{9 \cdot 25} = \sqrt{9} \cdot \sqrt{25} = 3 \cdot 5 = 15$, $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = 4$)
2. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ ($b \neq 0$) (e.g. $\sqrt{\frac{49}{4}} = \frac{\sqrt{49}}{\sqrt{4}} = \frac{7}{2}$, $\frac{\sqrt{72}}{\sqrt{8}} = \sqrt{\frac{72}{8}} = \sqrt{9} = 3$)
3. $(\sqrt{a})^n = \sqrt{a^n}$ (e.g. $(\sqrt{2})^{10} = \sqrt{2^{10}} = \sqrt{(2^5)^2} = 2^5 = 32$).

Proof

1. If $a \geq 0$ and $b \geq 0$ then there exist two positive real numbers x and y such that $x^2 = a$ and $y^2 = b$. So

$$x^2 \cdot y^2 = a \cdot b$$

$$(x \cdot y)^2 = a \cdot b$$

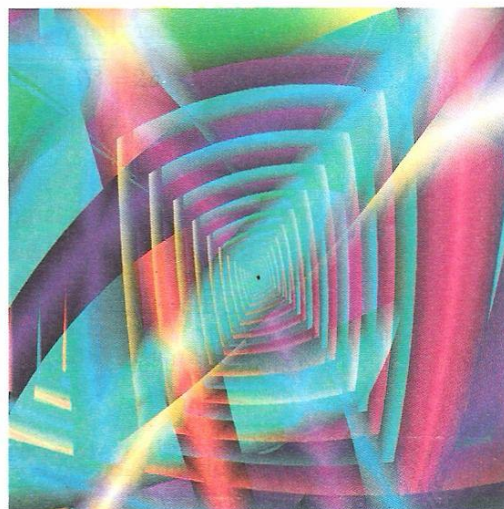
$$x \cdot y = \sqrt{a \cdot b}.$$

On the other hand,

$$x = \sqrt{a} \text{ and } y = \sqrt{b}$$

$$x \cdot y = \sqrt{a} \cdot \sqrt{b}.$$

Since $x \cdot y = \sqrt{a \cdot b}$ and $x \cdot y = \sqrt{a} \cdot \sqrt{b}$, we get $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.



2. This proof is similar to the proof of property 1. It is left as an exercise for you.
3. If $n \in \mathbb{Z}$ then either $n = 0$ or $n > 0$ or $n < 0$. Let us consider each case separately.

If $n = 0$ then $(\sqrt{a})^0 = 1$ and $\sqrt{a^0} = \sqrt{1} = 1$.

If $n > 0$ then $(\sqrt{a})^n = \underbrace{\sqrt{a} \cdot \sqrt{a} \cdot \sqrt{a} \cdots \sqrt{a}}_{n \text{ times}} = \underbrace{\sqrt{a \cdot a \cdot a \cdots a}}_{n \text{ times}} = \sqrt{a^n}.$

If $n < 0$ then $(\sqrt{a})^n = \frac{1}{(\sqrt{a})^{-n}} = \frac{1}{\underbrace{\sqrt{a} \cdot \sqrt{a} \cdots \sqrt{a}}_{-n \text{ times}}} = \frac{1}{\underbrace{\sqrt{a \cdot a \cdots a}}_{-n \text{ times}}} = \frac{1}{\sqrt{a^{-n}}} = \sqrt{\frac{1}{a^{-n}}} = \sqrt{a^n}.$

In conclusion, for all $n \in \mathbb{Z}$ we have $(\sqrt{a})^n = \sqrt{a^n}$.

EXAMPLE 18 Evaluate the expressions.

a. $\sqrt{\frac{36}{25}}$

b. $\sqrt{50} \cdot \sqrt{2}$

c. $\frac{\sqrt{24}}{\sqrt{6}}$

d. $\sqrt{25^4}$

Solution a. By the second property of square roots, $\sqrt{\frac{36}{25}} = \frac{\sqrt{36}}{\sqrt{25}} = \frac{6}{5}$.b. We cannot evaluate $\sqrt{50}$ or $\sqrt{2}$ exactly, but by using the first property of square roots we can calculate their exact product:

$$\sqrt{50} \cdot \sqrt{2} = \sqrt{50 \cdot 2} = \sqrt{100} = 10.$$

c. By property 2, $\frac{\sqrt{24}}{\sqrt{6}} = \sqrt{\frac{24}{6}} = \sqrt{4} = 2$.d. By property 3, $\sqrt{25^4} = \sqrt{(25^2)^2} = (\sqrt{25^2})^2 = 25^2 = 625$.

Remember that we can only apply the properties when the radicands are non-negative.

EXAMPLE 19 Evaluate $\sqrt{576}$.**Solution** Since we have a large number it is difficult to guess the required value. Instead, let us find the prime factorization of 576 and then use the properties of square roots:

$$\sqrt{576} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} = \sqrt{2^6 \cdot 3^2} = \sqrt{(2^3 \cdot 3)^2} = 2^3 \cdot 3 = 24.$$

We can verify the result by calculating $24^2 = 576$.**EXAMPLE 20** Evaluate each expression.

a. $\sqrt{14400}$

b. $\sqrt{3\frac{1}{4}} \cdot \sqrt{4\frac{17}{25}}$

c. $\sqrt{3} \cdot \sqrt{13} \cdot \sqrt{39}$

d. $\frac{\sqrt{2^3 \cdot 5^4 \cdot 7^3}}{\sqrt{7 \cdot 2^5 \cdot 5^6}}$

Solution a. $\sqrt{14400} = \sqrt{144 \cdot 100} = \sqrt{144} \cdot \sqrt{100} = 12 \cdot 10 = 120$

b. $\sqrt{3\frac{1}{4}} \cdot \sqrt{4\frac{17}{25}} = \sqrt{\frac{13}{4}} \cdot \sqrt{\frac{117}{25}} = \sqrt{\frac{13 \cdot 117}{4 \cdot 25}} = \frac{\sqrt{13 \cdot 13 \cdot 9}}{\sqrt{4 \cdot 25}} = \frac{\sqrt{13 \cdot 13} \cdot \sqrt{9}}{\sqrt{4} \cdot \sqrt{25}} = \frac{13 \cdot 3}{2 \cdot 5} = \frac{39}{10}$

c. $\sqrt{3} \cdot \sqrt{13} \cdot \sqrt{39} = \sqrt{3 \cdot 13} \cdot \sqrt{39} = \sqrt{39} \cdot \sqrt{39} = 39$

d. $\frac{\sqrt{2^3 \cdot 5^4 \cdot 7^3}}{\sqrt{7 \cdot 2^5 \cdot 5^6}} = \sqrt{\frac{2^3 \cdot 5^4 \cdot 7^3}{7 \cdot 2^5 \cdot 5^6}} = \sqrt{\frac{7^2}{2^2 \cdot 5^2}} = \sqrt{\left(\frac{7}{2 \cdot 5}\right)^2} = \frac{7}{2 \cdot 5} = \frac{7}{10}$



We often use the second and third properties of square roots together to remove perfect roots from the radicand. For example,

$$\begin{aligned}\sqrt{8} &= \sqrt{2^3} = \sqrt{2^2 \cdot 2} = \sqrt{2^2} \cdot \sqrt{2} = 2\sqrt{2} \text{ and} \\ \sqrt{360} &= \sqrt{2^3 \cdot 3^2 \cdot 5} = \sqrt{(2 \cdot 3)^2 \cdot 2 \cdot 5} = \sqrt{(2 \cdot 3)^2} \cdot \sqrt{2 \cdot 5} = 6\sqrt{10}.\end{aligned}$$

SIMPLIFYING A SQUARE ROOT

$$\sqrt{a^2b} = a\sqrt{b} \quad (a \geq 0, b \geq 0)$$

EXAMPLE 21 Simplify the square roots.

a. $\sqrt{80}$ b. $\sqrt{108}$ c. $\sqrt{1250}$

Solution

$$\begin{aligned}\text{a. } \sqrt{80} &= \sqrt{2^4 \cdot 5} = \sqrt{(2^2)^2 \cdot 5} = \sqrt{(2^2)^2} \cdot \sqrt{5} = 2^2 \cdot \sqrt{5} = 4\sqrt{5} \\ \text{b. } \sqrt{108} &= \sqrt{2^2 \cdot 3^3} = \sqrt{(2^2 \cdot 3^2) \cdot 3} = \sqrt{(2 \cdot 3)^2 \cdot 3} = \sqrt{(2 \cdot 3)^2} \cdot \sqrt{3} = 6\sqrt{3} \\ \text{c. } \sqrt{1250} &= \sqrt{2 \cdot 5^4} = \sqrt{(5^2)^2 \cdot 2} = \sqrt{(5^2)^2} \cdot \sqrt{2} = 25\sqrt{2}\end{aligned}$$

EXAMPLE 22 Simplify each expression, given $a > 0$, $b < 0$, $c > 0$ and $d < 0$.

a. $\sqrt{a^2b^2}$ b. $\sqrt{12a^5b^6c}$ c. $\sqrt{\frac{27a^3b}{32c^2d^9}}$

Solution a. First of all, $\sqrt{a^2b^2} = \sqrt{(ab)^2}$.

We cannot write $\sqrt{(ab)^2} = ab$ directly because we are dealing with variables not numbers, and these variables may be negative. So $\sqrt{(ab)^2} = |ab|$.

We know that $a > 0$ and $b < 0$, so $ab < 0$. Therefore $|ab| = -ab$, and so $\sqrt{(ab)^2} = -ab$.

$$\begin{aligned}\text{b. } \sqrt{12a^5b^6c} &= \sqrt{2^2 \cdot 3 \cdot (a^2)^2 \cdot a \cdot (b^3)^2 \cdot c} = \sqrt{(2a^2b^3)^2 \cdot 3ac} = \sqrt{(2a^2b^3)^2} \cdot \sqrt{3ac} \\ &= |2a^2b^3| \cdot \sqrt{3ac} = -2a^2b^3\sqrt{3ac} \text{ since } 2a^2b^3 < 0.\end{aligned}$$

$$\begin{aligned}\text{c. } \sqrt{\frac{27a^3b}{32c^2d^9}} &= \sqrt{\frac{3^2 \cdot 3 \cdot a^2 \cdot a \cdot b}{(2^2)^2 \cdot 2 \cdot c^2 \cdot (d^3)^2 \cdot d}} = \sqrt{\left(\frac{3a}{2^2 \cdot c \cdot d^3}\right)^2 \cdot \frac{3ab}{2d}} = \sqrt{\left(\frac{3a}{2^2 \cdot c \cdot d^3}\right)^2} \cdot \sqrt{\frac{3ab}{2d}} \\ &= \left|\frac{3a}{4cd^3}\right| \cdot \sqrt{\frac{3ab}{2d}} = \frac{3a}{4cd^3} \cdot \sqrt{\frac{3ab}{2d}} \text{ since } \frac{3a}{4cd^3} > 0.\end{aligned}$$



EXAMPLE 23 Write each number in the form $\sqrt{x}$.

a. $3\sqrt{2}$

b. $6\sqrt{6}$

c. $2\sqrt{65}$

Solution Remember that $a\sqrt{b} = \sqrt{a^2b}$, so squaring the number outside the radical sign and putting it as a product into the radicand will give us the desired result.

a. $3\sqrt{2} = \sqrt{3^2 \cdot 2} = \sqrt{18}$

b. $6\sqrt{6} = \sqrt{6^2 \cdot 6} = \sqrt{216}$

c. $2\sqrt{65} = \sqrt{2^2 \cdot 65} = \sqrt{260}$

EXAMPLE 24 Simplify $\sqrt{a^5 \cdot \sqrt{a^5} \cdot \sqrt{a^{10}}}$ if $a > 0$.

Solution To simplify expressions containing square roots inside each other, we begin by simplifying the innermost radical and proceed outwards. Note that since $a > 0$, we can apply the necessary properties without worrying about the absolute value of a .

$$\sqrt{a^5 \cdot \sqrt{a^5} \cdot \sqrt{a^{10}}} = \sqrt{a^5 \cdot \sqrt{a^5} \cdot a^5} = \sqrt{a^5 \cdot \sqrt{a^{10}}} = \sqrt{a^5 \cdot a^5} = \sqrt{a^{10}} = a^5$$

Theorem

For all $a \geq 0$ and $b \geq 0$,

$$a < b \text{ if and only if } \sqrt{a} < \sqrt{b}.$$

For example, $2 < 3$ means $\sqrt{2} < \sqrt{3}$ and $\sqrt{9} > \sqrt{8}$ means $9 > 8$.

Proof Since the numbers a and b are non-negative real numbers, $\sqrt{a}$ and $\sqrt{b}$ are also real numbers. If $a < b$ then

$$\begin{aligned} a - b &< 0 \\ (\sqrt{a})^2 - (\sqrt{b})^2 &< 0 \\ (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) &< 0 \\ \sqrt{a} - \sqrt{b} &< 0 && \text{(since } \sqrt{a} + \sqrt{b} \text{ is non-negative)} \\ \sqrt{a} &< \sqrt{b}. \end{aligned}$$

Conversely we can also prove that if $\sqrt{a} < \sqrt{b}$ then $a < b$. This part of the proof is left as an exercise for you.

EXAMPLE 25 Simplify the expression $\sqrt{(2 - \sqrt{5})^2}$.

Solution Remember that we are looking for the principal (positive) square root. Therefore we cannot assume that $\sqrt{(2 - \sqrt{5})^2} = 2 - \sqrt{5}$, because $2 - \sqrt{5}$ may be a negative number. Let us first use the theorem to check the sign of $2 - \sqrt{5}$.

Since $4 < 5$, the theorem tells us that $2 < \sqrt{5}$. So $2 - \sqrt{5} < 0$. This means that we must use the absolute value of $2 - \sqrt{5}$, to ensure that we find the principal root.

$$\text{Therefore, } \sqrt{(2 - \sqrt{5})^2} = |2 - \sqrt{5}| = -(2 - \sqrt{5}) = \sqrt{5} - 2.$$



EXAMPLE**26**

Determine whether each statement is true or false.

a. $\sqrt{5} > \sqrt{7}$

b. $\sqrt{20} < 2\sqrt{10}$

c. $4\sqrt{5} < 2\sqrt{20}$

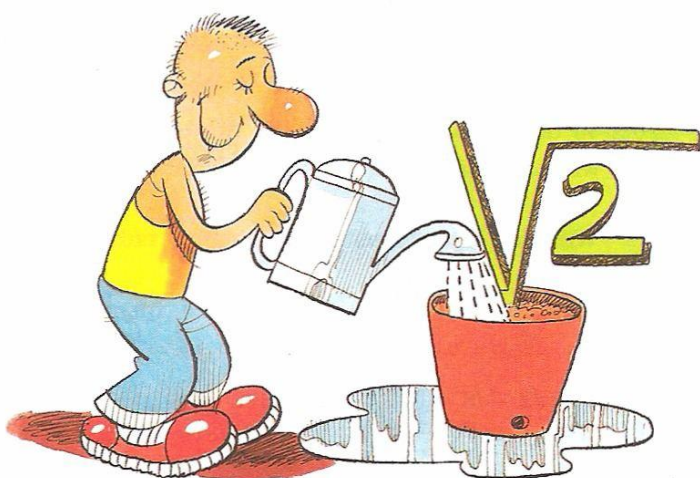
Solution

Let us square both sides of each inequality and use the theorem.

a. $(\sqrt{5})^2 = 5$ and $(\sqrt{7})^2 = 7$. If $5 < 7$ then $\sqrt{5} < \sqrt{7}$. So the statement is false.

b. $(\sqrt{20})^2 = 20$ and $(2\sqrt{10})^2 = 40$. If $20 < 40$ then $\sqrt{20} < 2\sqrt{10}$. So the statement is true.

c. $(4\sqrt{5})^2 = 80$ and $(2\sqrt{20})^2 = 80$. If $80 = 80$ then $4\sqrt{5} = 2\sqrt{20}$. So the statement is false.

**Check Yourself 6**

1. Evaluate each expression. Leave a square root in your answer if necessary.

a. $\frac{\sqrt{243}}{\sqrt{3}}$

b. $\sqrt{(1-\sqrt{2})^2}$

c. $\sqrt{3} \cdot \sqrt{21} \cdot \sqrt{7}$

d. $\frac{\sqrt{2^5 \cdot 7 \cdot 3^{11}}}{\sqrt{2^3 \cdot 7^5 \cdot 3^9}}$

2. Simplify the square roots.

a. $\sqrt{18}$

b. $\sqrt{120}$

3. Simplify each expression, given that $a < 0$, $b > 0$ and $c > 0$.

a. $\sqrt{20a^2b^3c}$

b. $\sqrt{\frac{2a^3b^7}{36ac^2}}$

Answers

1. a. 9

b. $\sqrt{2} - 1$

c. 21

d. $\frac{6}{49}$

2. a. $3\sqrt{2}$

b. $2\sqrt{30}$

3. a. $-2ab\sqrt{5bc}$

b. $-\frac{ab^3}{3c}\sqrt{\frac{b}{2}}$

C. WORKING WITH SQUARE ROOTS

1. Addition and Subtraction

Recall that a radical expression is an expression which contains a radical sign. The radicand is the number below the radical sign in an expression. For example, 5 is the radicand in the radical expression $\sqrt{5}$. Unfortunately, there is no easy way to add or subtract radical expressions with different radicands. In other words, it is usually difficult to simplify a sum such as $\sqrt{a} + \sqrt{b}$ or a difference such as $\sqrt{a} - \sqrt{b}$ if $a \neq b$.

To simplify the sum $a\sqrt{b} + c\sqrt{b}$, we can apply the distributive property to the common factor $\sqrt{b}$. Thus, $a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$. But we cannot combine terms which do not have the same radicand: $2\sqrt{2} + 3\sqrt{3} + 5\sqrt{2} = 7\sqrt{2} + 3\sqrt{3}$. We cannot simplify this expression further.

To determine whether or not two or more terms have the same radicand, we begin by writing each term in its simplest form.

EXAMPLE 27

Write the expressions in their simplest radical form.

a. $3\sqrt{3} + \sqrt{2} - 5\sqrt{3}$ b. $5\sqrt{2} - \sqrt{8}$ c. $\sqrt{27} - \sqrt{48}$ d. $\sqrt{50} + \sqrt{48} - \sqrt{72} + \sqrt{75}$

Solution

a. $3\sqrt{3} + \sqrt{2} - 5\sqrt{3} = (3 - 5)\sqrt{3} + \sqrt{2} = -2\sqrt{3} + \sqrt{2}$

b. $5\sqrt{2} - \sqrt{8} = 5\sqrt{2} - 2\sqrt{2} = 3\sqrt{2}$

c. $\sqrt{27} - \sqrt{48} = 3\sqrt{3} - 4\sqrt{3} = -\sqrt{3}$

d. $\sqrt{50} + \sqrt{48} - \sqrt{72} + \sqrt{75} = 5\sqrt{2} + 4\sqrt{3} - 6\sqrt{2} + 5\sqrt{3} = -\sqrt{2} + 9\sqrt{3}$.

EXAMPLE 28

Evaluate the sum $(\sqrt{3})^6 - (\sqrt{5})^5 - (\sqrt{2})^8 + (\sqrt{5})^3$ in radical form.

Solution

$$\begin{aligned}(\sqrt{3})^6 - (\sqrt{5})^5 - (\sqrt{2})^8 + (\sqrt{5})^3 &= \sqrt{3^6} - \sqrt{5^5} - \sqrt{2^8} + \sqrt{5^3} = \sqrt{(3^3)^2} - \sqrt{(5^2)^2 \cdot 5} - \sqrt{(2^4)^2} + \sqrt{5^2 \cdot 5} \\&= 3^3 - 5^2\sqrt{5} - 2^4 + 5\sqrt{5} = 11 - 20\sqrt{5}\end{aligned}$$

Common Mistakes

$$\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$$

$$\sqrt{a^2 + b^2} \neq a + b$$

$$(\sqrt{a} + \sqrt{b})^2 \neq a + b$$

2. Multiplication and Division

To multiply or divide radical expressions, we can use the general properties of square roots ($\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$). We can also use the distributive property of multiplication over addition and subtraction to simplify the products of expressions that contain radicals.

EXAMPLE 29 Evaluate each expression. Keep your answer in radical form if necessary.

a. $3\sqrt{12} \cdot 2\sqrt{3}$ b. $(2 + \sqrt{6})(3\sqrt{2} - 2\sqrt{3})$ c. $(12\sqrt{45} - 6\sqrt{20}) \div 3\sqrt{5}$

Solution a. $3\sqrt{12} \cdot 2\sqrt{3} = 3 \cdot 2 \cdot \sqrt{12 \cdot 3} = 6\sqrt{36} = 6 \cdot 6 = 36$

b. $(2 + \sqrt{6})(3\sqrt{2} - 2\sqrt{3}) = 2 \cdot 3\sqrt{2} - 2 \cdot 2\sqrt{3} + \sqrt{6} \cdot 3\sqrt{2} - \sqrt{6} \cdot 2\sqrt{3} = 6\sqrt{2} - 4\sqrt{3} + 3\sqrt{12} - 2\sqrt{18}$
 $= 6\sqrt{2} - 4\sqrt{3} + 3 \cdot 2\sqrt{3} - 2 \cdot 3\sqrt{2} = 6\sqrt{2} - 4\sqrt{3} + 6\sqrt{3} - 6\sqrt{2} = 2\sqrt{3}$

c. $(12\sqrt{45} - 6\sqrt{20}) \div 3\sqrt{5} = \frac{12\sqrt{45}}{3\sqrt{5}} - \frac{6\sqrt{20}}{3\sqrt{5}} = \frac{12}{3} \sqrt{\frac{45}{5}} - \frac{6}{3} \sqrt{\frac{20}{5}} = 4\sqrt{9} - 2\sqrt{4} = 4 \cdot 3 - 2 \cdot 2 = 8$

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EXAMPLE 30 Evaluate $[(\frac{4}{3}\sqrt{3} + \sqrt{2} + \sqrt{3\frac{1}{3}}) + (-\sqrt{\frac{1}{3}})] + \sqrt{2}(\sqrt{3} + \sqrt{5})$.

Solution $[(\frac{4}{3}\sqrt{3} + \sqrt{2} + \sqrt{3\frac{1}{3}}) + (-\sqrt{\frac{1}{3}})] + \sqrt{2}(\sqrt{3} + \sqrt{5}) = [(\frac{4}{3}\sqrt{3} + \sqrt{2} + \sqrt{\frac{10}{3}}) + (-\sqrt{\frac{1}{3}})] + \sqrt{2}(\sqrt{3} + \sqrt{5})$
 $= -4 - \sqrt{6} - \sqrt{10} + \sqrt{6} + \sqrt{10} = -4$

EXAMPLE 31 Compare $\sqrt{6} + 1$ and $\sqrt{5} + \sqrt{2}$.

Solution Let us square both expressions.

$(\sqrt{6} + 1)^2 = (\sqrt{6} + 1)(\sqrt{6} + 1) = 7 + 2\sqrt{6}$ and $(\sqrt{5} + \sqrt{2})^2 = (\sqrt{5} + \sqrt{2})(\sqrt{5} + \sqrt{2}) = 7 + 2\sqrt{10}$.
 Since $(7 + 2\sqrt{6}) < (7 + 2\sqrt{10})$, we can conclude that $(\sqrt{6} + 1)^2 < (\sqrt{5} + \sqrt{2})^2$ and so $(\sqrt{6} + 1) < (\sqrt{5} + \sqrt{2})$.

Check Yourself 7

1. Evaluate $\sqrt{18} - \sqrt{50} + \sqrt{12} + \sqrt{8}$.
2. Evaluate $(\frac{1}{2}\sqrt{6} - 3\sqrt{3} + 5\sqrt{2} - \sqrt{8})\sqrt{24} + 18\sqrt{2} - 12\sqrt{3}$.
3. Compare $\sqrt{6} - \sqrt{2}$ and $\sqrt{5} - \sqrt{3}$.

Answers

1. $2\sqrt{3}$
2. 6
3. $(\sqrt{5} - \sqrt{3}) < (\sqrt{6} - \sqrt{2})$



3. Rationalizing a Denominator

Look at the fractions $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3}}{\sqrt{5}}$ and $\frac{4}{\sqrt{12}}$. These fractions have a common property: their denominators are irrational. In mathematics, it is usually easier to work with fractions which have rational denominators. Therefore, if a fraction has an irrational denominator it is usually better to change it to a rational number. This is called **rationalizing the denominator** of the fraction. We rationalize the denominator by multiplying both the numerator and the denominator of the fraction by a suitable number so that the denominator becomes rational. For example, if the fraction is in the form $\frac{a}{\sqrt{b}}$ we multiply both the numerator and denominator by $\sqrt{b}$:

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a \cdot \sqrt{b}}{\sqrt{b} \cdot \sqrt{b}} = \frac{a\sqrt{b}}{b}$$

↑
rational denominator

Notice that the value of the fraction is still the same because we multiplied it by a fraction equal to 1.

EXAMPLE 32

Rationalize the denominator of each fraction.

a. $\frac{4}{\sqrt{2}}$

b. $\frac{\sqrt{7}}{\sqrt{5}}$

c. $\frac{7\sqrt{3}}{\sqrt{14}}$

d. $\frac{3\sqrt{6}}{\sqrt{18}}$

Solution a. $\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

b. $\frac{\sqrt{7}}{\sqrt{5}} = \frac{\sqrt{7}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{35}}{5}$

c. $\frac{7\sqrt{3}}{\sqrt{14}} = \frac{7\sqrt{3}}{\sqrt{14}} \cdot \frac{\sqrt{14}}{\sqrt{14}} = \frac{7\sqrt{42}}{14} = \frac{\sqrt{42}}{2}$

d. $\frac{3\sqrt{6}}{\sqrt{18}} = \frac{3\sqrt{6}}{\sqrt{18}} \cdot \frac{\sqrt{18}}{\sqrt{18}} = \frac{3\sqrt{108}}{18} = \frac{3 \cdot 6\sqrt{3}}{18} = \frac{18\sqrt{3}}{18} = \sqrt{3}$

Alternatively, $\frac{3\sqrt{6}}{\sqrt{18}} = 3 \cdot \frac{\sqrt{6}}{\sqrt{18}} = 3 \cdot \sqrt{\frac{1}{3}} = 3 \cdot \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$.



In the previous example the denominators of each fraction contained just one term. The procedure for rationalizing a denominator which contains more than one term is a little different. For example, if the fraction is in the form $\frac{1}{\sqrt{a}-\sqrt{b}}$ we multiply both the numerator and denominator by $\sqrt{a}+\sqrt{b}$. Look at an example:

$$\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{1}{\sqrt{a}-\sqrt{b}} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})} = \frac{\sqrt{a}+\sqrt{b}}{a-b}.$$



Algebraic identities:

$$(x-y)(x+y) = x^2 - y^2$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

Notice that the original denominator is $\sqrt{a}-\sqrt{b}$ and we multiply it by $\sqrt{a}+\sqrt{b}$ (not by $\sqrt{a}-\sqrt{b}$).

This means that we can use the identity $(x-y)(x+y) = x^2 - y^2$.

In other words,

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b, \text{ which has no irrational terms.}$$

Notice that if we multiply $\sqrt{a}-\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$ we obtain

$$(\sqrt{a}-\sqrt{b})^2 = (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 = a^2 - 2\sqrt{ab} + b^2, \text{ which still contains an irrational term.}$$

Definition

conjugate

An expression with exactly two terms is called a binomial expression. Two binomial expressions whose first terms are equal and last terms are opposite are called **conjugates**.

For example, $x+y$ and $x-y$ are conjugates. $3x-6$ and $3x+6$ are also conjugates.



To find the conjugate of a binomial, just change the sign of the second term.

We can conclude that multiplying an irrational binomial expression by its conjugate will turn it into a rational expression. For example,

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a - b$$

$$(\sqrt{3}-2)(\sqrt{3}+2) = 3 - 4 = -1$$

$$(-2\sqrt{2}+\sqrt{3})(-2\sqrt{2}-\sqrt{3}) = (-2\sqrt{2})^2 - (\sqrt{3})^2 = 8 - 3 = 5.$$

EXAMPLE

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Rationalize the denominator of each fraction.

a. $\frac{1}{\sqrt{3}+1}$

b. $\frac{2}{\sqrt{3}-\sqrt{2}}$

c. $\frac{\sqrt{5}+\sqrt{2}}{2\sqrt{5}-3\sqrt{2}}$

Solution

a. $\frac{1}{\sqrt{3}+1} = \frac{1 \cdot (\sqrt{3}-1)}{(\sqrt{3}+1) \cdot (\sqrt{3}-1)} = \frac{\sqrt{3}-1}{(\sqrt{3})^2 - 1^2} = \frac{\sqrt{3}-1}{2}$

b. $\frac{2}{\sqrt{3}-\sqrt{2}} = \frac{2(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{2\sqrt{3}+2\sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} = \frac{2\sqrt{3}+2\sqrt{2}}{1} = 2\sqrt{3}+2\sqrt{2}$

c. $\frac{\sqrt{5}+\sqrt{2}}{2\sqrt{5}-3\sqrt{2}} = \frac{(\sqrt{5}+\sqrt{2})(2\sqrt{5}+3\sqrt{2})}{(2\sqrt{5}-3\sqrt{2})(2\sqrt{5}+3\sqrt{2})} = \frac{2 \cdot 5 + 3\sqrt{10} + 2\sqrt{10} + 3 \cdot 2}{(2\sqrt{5})^2 - (3\sqrt{2})^2} = \frac{16+5\sqrt{10}}{2}$



EXAMPLE

34

Rationalize the denominator of each fraction.

a. $\frac{1}{\sqrt{\sqrt{6}+2}}$

b. $\frac{1}{\sqrt{3}+\sqrt{2}+\sqrt{5}}$

Solution

a. $\frac{1}{\sqrt{\sqrt{6}+2}} = \frac{\sqrt{\sqrt{6}-2}}{\sqrt{\sqrt{6}+2} \cdot \sqrt{\sqrt{6}-2}} = \frac{\sqrt{\sqrt{6}-2}}{\sqrt{(\sqrt{6})^2 - (\sqrt{2})^2}} = \frac{\sqrt{\sqrt{6}-2}}{\sqrt{6-2}} = \frac{\sqrt{\sqrt{6}-2}}{2}$

b. First we write $\sqrt{3} + \sqrt{2} + \sqrt{5}$ as $(\sqrt{3} + \sqrt{2}) + \sqrt{5}$.

The conjugate of $(\sqrt{3} + \sqrt{2}) + \sqrt{5}$ is $(\sqrt{3} + \sqrt{2}) - \sqrt{5}$.

$$\begin{aligned} \frac{1}{\sqrt{3}+\sqrt{2}+\sqrt{5}} &= \frac{\sqrt{3}+\sqrt{2}-\sqrt{5}}{(\sqrt{3}+\sqrt{2}+\sqrt{5})(\sqrt{3}+\sqrt{2}-\sqrt{5})} = \frac{\sqrt{3}+\sqrt{2}-\sqrt{5}}{(\sqrt{3}+\sqrt{2})^2 - (\sqrt{5})^2} \\ &= \frac{\sqrt{3}+\sqrt{2}-\sqrt{5}}{3+2\sqrt{6}+2-5} = \frac{\sqrt{3}+\sqrt{2}-\sqrt{5}}{2\sqrt{6}} = \frac{(\sqrt{3}+\sqrt{2}-\sqrt{5}) \cdot \sqrt{6}}{2\sqrt{6} \cdot \sqrt{6}} = \frac{3\sqrt{2}+2\sqrt{3}-\sqrt{30}}{12} \end{aligned}$$

EXAMPLE

35

Write $\sqrt{3-2\sqrt{2}} + \sqrt{3+2\sqrt{2}}$ in its simplest form.

Solution

Note that $3 - 2\sqrt{2}$ and $3 + 2\sqrt{2}$ are conjugates of each other.

Let $\sqrt{3-2\sqrt{2}} + \sqrt{3+2\sqrt{2}} = x$.

$$\begin{aligned} \text{Squaring both sides, } x^2 &= 3-2\sqrt{2} + 2\sqrt{3-2\sqrt{2}} \cdot \sqrt{3+2\sqrt{2}} + 3+2\sqrt{2} \\ &= 6 + 2\sqrt{3^2 - (2\sqrt{2})^2} = 6 + 2\sqrt{9-8} = 8. \end{aligned}$$

So $x = \sqrt{8}$ or $x = -\sqrt{8}$. Notice that x is the sum of two principal (positive) square roots, so x must be positive. So the result is $\sqrt{8}$.

EXAMPLE

36

Simplify $\frac{x\sqrt{x}-\sqrt{x}}{x-\sqrt{x}}$.

Solution 1

$$\begin{aligned} \frac{x\sqrt{x}-\sqrt{x}}{x-\sqrt{x}} &= \frac{(x\sqrt{x}-\sqrt{x})(x+\sqrt{x})}{(x-\sqrt{x})(x+\sqrt{x})} = \frac{x^2\sqrt{x}+x^2-x\sqrt{x}-x}{x^2-x} = \frac{x\sqrt{x} \cdot (x-1) + x(x-1)}{x(x-1)} \\ &= \frac{x(x-1)(\sqrt{x}+1)}{x(x-1)} = \sqrt{x}+1 \end{aligned}$$

Solution 2

$$\frac{x\sqrt{x}-\sqrt{x}}{x-\sqrt{x}} = \frac{x \cdot \sqrt{x} - \sqrt{x}}{\sqrt{x} \cdot \sqrt{x} - \sqrt{x}} = \frac{\sqrt{x} \cdot (x-1)}{\sqrt{x} \cdot (\sqrt{x}-1)} = \frac{(\sqrt{x})^2 - 1^2}{\sqrt{x}-1} = \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \sqrt{x}+1$$



4. Finding the Square Root of an Irrational Sum

The following formula helps us to simplify expressions of the form $\sqrt{a-\sqrt{b}}$ or $\sqrt{a+\sqrt{b}}$. You can verify the formula by squaring each side and simplifying.

SQUARE ROOT OF AN IRRATIONAL SUM

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

EXAMPLE 37 Simplify $\sqrt{4+\sqrt{7}}$.

Solution
$$\sqrt{4+\sqrt{7}} = \sqrt{\frac{4+\sqrt{4^2-7}}{2}} + \sqrt{\frac{4-\sqrt{4^2-7}}{2}} = \sqrt{\frac{4+\sqrt{9}}{2}} + \sqrt{\frac{4-\sqrt{9}}{2}} = \sqrt{\frac{4+3}{2}} + \sqrt{\frac{4-3}{2}} = \sqrt{\frac{7}{2}} + \sqrt{\frac{1}{2}}$$

EXAMPLE 38 Simplify $\sqrt{17-4\sqrt{9+4\sqrt{5}}}$.

Solution
$$\begin{aligned}\sqrt{17-4\sqrt{9+4\sqrt{5}}} &= \sqrt{17-4\sqrt{9+\sqrt{80}}} = \sqrt{17-4\left(\sqrt{\frac{9+\sqrt{9^2-80}}{2}} + \sqrt{\frac{9-\sqrt{9^2-80}}{2}}\right)} \\ &= \sqrt{17-4(\sqrt{5}+2)} = \sqrt{9-4\sqrt{5}} = \sqrt{\frac{9+\sqrt{9^2-80}}{2}} - \sqrt{\frac{9-\sqrt{9^2-80}}{2}} = \sqrt{5} - 2\end{aligned}$$

EXAMPLE 39 Simplify $\sqrt{6+\sqrt{6+\sqrt{6+\dots}}}$.

Solution Let $\sqrt{6+\sqrt{6+\sqrt{6+\dots}}} = x$. Squaring both sides gives us $6 + \sqrt{6+\sqrt{6+\dots}} = x^2$, i.e. $6 + x = x^2$. If we solve the quadratic equation $6 + x = x^2$, the solution is $x = -2$ or $x = 3$. Since we are looking for $x = \sqrt{6+\sqrt{6+\sqrt{6+\dots}}} > 0$, the answer is 3.

Check Yourself 8

1. Rationalize the denominator of each fraction.

a. $\frac{6}{\sqrt{12}}$

b. $\frac{1+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

c. $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{5}+\sqrt{3}}$

2. Simplify $\frac{\sqrt{x+x}}{\sqrt{x+1}}$.

3. Simplify $\sqrt{3+2\sqrt{2}}$.

Answers

1. a. $\sqrt{3}$ b. $\sqrt{6} + \sqrt{3} - \sqrt{2} - 2$ c. $\sqrt{15} - 4$ 2. $\sqrt{x}$ 3. $\sqrt{2} + 1$

CALCULATING SQUARE ROOTS

We can calculate the square root of any number with the help of a calculator. But is there a formula for finding the square root? In fact, there are several ways to find the square root of a number, depending on the accuracy we want. Here are two possible methods.

Method 1:

To approximate the square root of x (for example, the square root of 21):

1. Subtract the closest perfect square which is smaller than x ($21 - 16 = 5$).
2. Divide the result by the square root of this perfect square ($\sqrt{16}$ is 4, so $5 \div 4 = 1.25$).
3. Divide the result by 2 ($1.25 \div 2 = 0.625$).
4. Add the result to the square root of the perfect square ($4 + 0.625 = 4.625$).

This is the approximate square root of x .

The actual square root is $\sqrt{21} = 4.582575695\dots$. Although our result is not perfect, we have a close approximation to the real value.

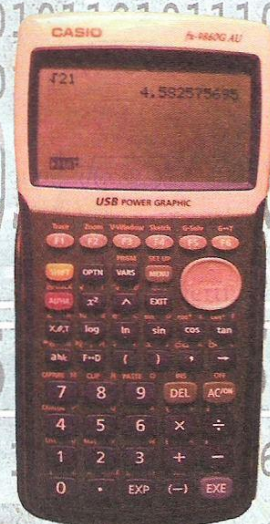
Method 2:

To approximate the square root of x (for example, the square root of 21):

1. Make a guess (for example, 4.625).
2. Divide x by your guess ($21 \div 4.625 = 4.5405\dots$).
3. Find the average of your guess and the result of step 2 ($(4.625 + 4.5405) \div 2 = 4.5827\dots$).
4. Use this result as your new 'guess' and repeat steps 2 and 3 ($21 \div 4.5827 = 4.5823\dots$ and the average of 4.5827 and 4.5823 is 4.5825...).

Keeping in mind that the actual square root is 4.582575695..., we have an almost perfect approximation. We can repeat step 4 if we need a more accurate result. This method is surprisingly fast, even when you make a really bad guess.

Think about the logic behind these two methods. Why did we divide the result by 2 in step 3 of the first method? Why did we calculate the average in the second method? Try to find different ways of calculating square roots.



EXERCISES 2.2

A. Understanding Square Roots

- Find the principal square root of each number.
 - 36
 - 484
 - $\frac{36}{25}$
 - $\frac{121}{81}$
 - $\frac{25}{400}$
 - 0.0004
 - 0.0049
 - 0.25
- If possible, evaluate the square root exactly. Otherwise, approximate the result to two decimal places.
 - $-\sqrt{81}$
 - $\sqrt{576}$
 - $-\sqrt{0.09}$
 - $\sqrt{3}$
 - $-\sqrt{\frac{36}{49}}$
 - $\sqrt{\frac{169}{81}}$
- Approximate each square root to two decimal places using a calculator.
 - $\sqrt{40}$
 - $\sqrt{\frac{3}{5}}$
 - $\sqrt{7}$
 - $\sqrt{8}$
 - $\sqrt{0.2}$
 - $\sqrt{\frac{27}{2}}$
- Simplify $\sqrt{a^2} + \sqrt{b^2} - a - b$ if $a < 0$ and $b > 0$.
- Simplify $\sqrt{(x+3)^2} + \sqrt{x^2}$ if $-3 < x < 0$.
- Simplify $\sqrt{x^2} + \sqrt{m^2} + \sqrt{1-2y+y^2}$ if $x < 0$ and $1 < y < m$.
- Simplify $\sqrt{(a-2)^2} + \sqrt{(a-4)^2}$ if $2 < a < 4$.
- Simplify $\sqrt{a^2 - 13a + 45} + \sqrt{a^2 - 8a + 16}$ if $a < 4$.
- Find x and y which satisfy $\sqrt{x^2} + \sqrt{y^2} - 4y + 4 = 0$.

B. Properties of Square Roots

- Evaluate the expressions.
 - $\sqrt{10} \cdot \sqrt{160}$
 - $\sqrt{(\sqrt{7}-3)^2}$
 - $\frac{\sqrt{99}}{\sqrt{275}}$
 - $\sqrt{2\frac{5}{7}} \cdot \sqrt{\frac{7}{19}}$
 - $\sqrt{9^3}$
 - $\sqrt{2.56}$
- $x < 0$, $y > 0$ and $z < 0$ are given. Simplify the expressions.
 - $\sqrt{x^2 y^2}$
 - $\sqrt{x^4 y^6 z^4}$
 - $\sqrt{(yz)^2}$
 - $\sqrt{x^2 z^4}$
 - $\sqrt{(xyz)^4}$
 - $\sqrt{(xy)^2 \cdot (yz)^6}$
- Express each expression in its simplest radical form. ($x > 0$, $y > 0$)
 - $\frac{\sqrt{3x} \cdot \sqrt{5}}{\sqrt{27x}}$
 - $\frac{\sqrt{6x} \cdot \sqrt{8}}{\sqrt{2x}}$
 - $\sqrt{x^2 + y^2}$
 - $\sqrt{x^4 - 4x^2}$
 - $\sqrt{x^5 + 4x^4}$
 - $\sqrt{x^4} \cdot \sqrt{y^2}$
- Write each expression in its simplest form, given that all the variables represent positive real numbers.
 - $\sqrt{12x^3 y^2}$
 - $\sqrt{24x^5 y^2 z^3}$
 - $\frac{\sqrt{a^2 b^3 z^4 t^4}}{\sqrt{80x^2 y^2 z^2 t^2}}$
 - $\frac{\sqrt{18xz}}{\sqrt{98y^2 x^{-1} z^3}}$
- Write each expression in the form $\sqrt{x}$, given that all the variables represent positive real numbers.
 - $a\sqrt{a^3}$
 - $a^2 b \sqrt{ab}$
 - $3x^4 \sqrt{\frac{x}{3}}$
 - $(b+4)\sqrt{\frac{5}{b^3+8b+16}}$, if $b < -4$

15. Without using a calculator, determine whether each number is rational or irrational.

a. $\sqrt{74529}$ b. $\sqrt{237584}$
 c. $\sqrt{\sqrt{138384} - 9}$ d. $\sqrt{287 - \sqrt{20449}}$

16. Determine whether each statement is true or false.

a. $2\sqrt{5} < \sqrt{21}$ b. $-3\sqrt{5} > -5\sqrt{2}$

C. Working with Square Roots

17. Simplify each operation by combining as many terms as possible. All variables represent positive real numbers.

a. $2\sqrt{3} + 3\sqrt{3}$ b. $5\sqrt{2} - 4\sqrt{2}$
 c. $-5\sqrt{x} + 2\sqrt{x}$ d. $10\sqrt{x} - 3\sqrt{x}$
 e. $2\sqrt{x} + \sqrt{x} + 5\sqrt{x}$ f. $\sqrt{3} - 2\sqrt{2} + 5\sqrt{3}$
 g. $\sqrt{2} - \sqrt{8}$ h. $\sqrt{18} - 3\sqrt{2}$
 i. $2\sqrt{27} + 2\sqrt{12}$ j. $\sqrt{4x} - \sqrt{16x}$
 k. $\sqrt{8xy} + 2\sqrt{18xy}$ l. $\sqrt{\frac{3}{2}} - \sqrt{\frac{2}{3}}$
 m. $\sqrt{\frac{1}{8}} + \sqrt{8}$ n. $\sqrt{\frac{ab}{2}} + \sqrt{8ab}$
 o. $\sqrt{\frac{1}{2}} - \sqrt{18}$ p. $\frac{\sqrt{3}}{3} + 3\sqrt{\frac{1}{3}} + \sqrt{18}$

18. Simplify the operations.

a. $\sqrt{27} - \sqrt{48} - \sqrt{108} + \sqrt{75}$
 b. $\sqrt{45} + \sqrt{80} - 2\sqrt{20} - \sqrt{125}$
 c. $\sqrt{98} + 3\sqrt{32} - 2\sqrt{50} + 4\sqrt{8} - 2\sqrt{72}$
 d. $2\sqrt{\frac{15}{2}} - 6\sqrt{\frac{10}{3}}$
 e. $\sqrt{x^2y} + \sqrt{z^2y} + x\sqrt{y} + z\sqrt{y}$ ($x < 0, y > 0, z > 0$)

19. Evaluate each expression.

a. $3(\sqrt{3} + 1)$ b. $2(\sqrt{2} - 2)$
 c. $2(5 - \sqrt{2})$ d. $\sqrt{3}(\sqrt{3} + 4)$
 e. $\sqrt{6}(\sqrt{6} - 1)$ f. $\sqrt{x}(\sqrt{x} - 1)$
 g. $\sqrt{y}(5 - \sqrt{y})$ h. $\sqrt{m}(\sqrt{m} + \sqrt{n})$
 i. $\sqrt{6}(\sqrt{3} - 2)$ j. $\sqrt{3}(1 + \sqrt{6})$
 k. $(4\sqrt{x} - 2)(\sqrt{x} + 2)$ l. $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$
 m. $(\sqrt{2} - 1)(\sqrt{8} + 2)$ n. $(1 + \sqrt{2})^2(3 - 2\sqrt{2})$

20. Evaluate each expression.

a. $(\frac{1}{2}\sqrt{32} - \frac{1}{3}\sqrt{3} + 4\sqrt{15})2\sqrt{3} - 4\sqrt{6} - 24\sqrt{5}$
 b. $(\frac{1}{4} + \sqrt{\frac{1}{5}} - 2\sqrt{2}) + \frac{1}{2}\sqrt{\frac{1}{5}} - \frac{1}{2}\sqrt{5} + 4\sqrt{10}$

21. Determine whether each statement is true or false.

a. $(\sqrt{5} + \sqrt{3}) > (\sqrt{7} + 1)$
 b. $(\sqrt{10} - \sqrt{2}) < (3 - \sqrt{3})$
 c. $(3 - \sqrt{3})^2 < 2$
 d. $(\sqrt{2} + \sqrt{3})^2 < (\sqrt{3} + \sqrt{5})$

22. Rationalize the denominators.

a. $\frac{6}{\sqrt{2}}$ b. $\frac{3\sqrt{5}}{\sqrt{15}}$ c. $\frac{1}{\sqrt{3} + 1}$
 d. $\frac{4}{\sqrt{6} - 3}$ e. $\frac{\sqrt{x}}{\sqrt{x} - 4}$ f. $\frac{\sqrt{3} + 1}{\sqrt{3} - 1}$
 g. $\frac{\sqrt{a} - 3}{\sqrt{a} + 2}$ h. $\frac{1 + \sqrt{2}}{\sqrt{3} + \sqrt{2}}$ i. $\frac{\sqrt{5}}{\sqrt{5} + \sqrt{3}}$

23. Rationalize the denominators.

a. $\frac{1}{\sqrt{\sqrt{5} + 2}}$ b. $\frac{1}{\sqrt{\sqrt{a} + \sqrt{b}}}$
 c. $\frac{\sqrt{3}}{\sqrt{3} + 2\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{3} - 2\sqrt{2}}$ d. $\frac{1}{\sqrt{6} + \sqrt{3} + \sqrt{2}}$



24. Write each expression in its simplest form.

- a. $\sqrt{4+2\sqrt{3}}$ b. $\sqrt{5+2\sqrt{6}}$
 c. $\sqrt{6-4\sqrt{2}}$ d. $\sqrt{4-\sqrt{7}}$
 e. $\sqrt{4-2\sqrt{3}} + \sqrt{4+2\sqrt{3}}$

25. Simplify the expressions.

- a. $\sqrt{\sqrt{3}-1} \cdot \sqrt{\sqrt{3}+1} \cdot \sqrt{2}$
 b. $(\sqrt{5+\sqrt{2}} + \sqrt{3+\sqrt{2}})(\sqrt{5+\sqrt{2}} - \sqrt{3+\sqrt{2}})$
 c. $(\sqrt{\sqrt{a}+\sqrt{b}} + \sqrt{\sqrt{a}-\sqrt{b}})(\sqrt{\sqrt{a}+\sqrt{b}} - \sqrt{\sqrt{a}-\sqrt{b}})$

26. Evaluate each expression.

- a. $\sqrt{\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}} - \sqrt{\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}}$
 b. $\frac{1}{2+\sqrt{5}} - \frac{1}{\sqrt{7}+3} + \frac{3}{1-\sqrt{7}} - \frac{10}{\sqrt{5}} + \sqrt{5}$
 c. $\sqrt{2-\sqrt{2-\sqrt{2}-\dots}}$
 d. $\sqrt{6+2\sqrt{5-\sqrt{13+\sqrt{48}}}}$

Mixed Problems

27. Evaluate

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \dots + \frac{1}{\sqrt{1000}+\sqrt{1001}}$$

28. Order $(\sqrt{2} + \sqrt{11})$, $(\sqrt{3} + \sqrt{10})$ and 5.

29. Compare $(\sqrt{5} + \sqrt{20} + \sqrt{60})$ and $(\sqrt{10} + \sqrt{30} + \sqrt{50})$.

30. Solve each equation.

- a. $\sqrt{x-3} = 5$ b. $\sqrt{x^2-2x} = \sqrt{x+4}$
 c. $\sqrt{x+2} = 2x-11$ d. $\sqrt{x^2-x+4} = 3x-8$

31. Simplify the expressions.

- a. $\frac{a}{\sqrt{ab}-b} + \frac{\sqrt{b}}{\sqrt{b}-\sqrt{a}}$
 b. $(\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}} + \frac{\sqrt{b}}{\sqrt{c}}) \div \frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}$
 c. $(\frac{a+1}{\sqrt{a}} + \frac{1}{a-\sqrt{a}} - \frac{a}{\sqrt{a}+1}) \cdot \frac{\sqrt{3}-a\sqrt{3}}{a+1}$
 d. $(1+\sqrt{1-a^2} + \frac{a^2}{\sqrt{1-a^2}}) \div (\frac{1}{1-a^2} + \frac{1}{\sqrt{1-a^2}})$

32. If $\sqrt{a} - \frac{1}{\sqrt{a}} = 2$, evaluate $a + \frac{1}{a}$.

33. $a\sqrt{b} = 6$ and $b\sqrt{a} = 18$ are given. Express b in terms of a .

34. Show that $3 - \sqrt{2}$ is not a solution of the equation $x^3 - 7x^2 - 7x + 1 = 0$.

35. Mustafacan wants to find the height of a house by flying his kite as shown in the figure. The length of the string from his hand to the kite is 50 m and he is holding the string 1 m above the ground. Given that Mustafacan is standing 30 m away from the house, use the Pythagorean Theorem to help Mustafacan find the height of the house.



3

RADICAL EXPRESSIONS AND RATIONAL EXPONENTS

A. RADICAL EXPRESSIONS

1. n th Root of a Number

Up to now we have studied exponential expressions of the form a^n , where n is any integer and a is a real number. We have learned how to evaluate numbers like 2^4 , 3^{-5} and 9^0 . But what do expressions such as $3^{1/2}$ and $8^{2/3}$ mean? In this section we will extend our study of exponents to include rational exponents. But before doing this, we need to understand the concept of n th root.

Square roots are a special case of n th roots for $n = 2$. More generally, the n th root of a is the number which gives a as a result when it is raised to the n th power.

Definition

principal n th root

Let a and b be real numbers, and let n be a natural number different from 1 such that $b^n = a$. Then b is called the **principal n th root** of a , denoted by $b = \sqrt[n]{a}$.

For example,

$$\sqrt[2]{9} = 3 \text{ ('3 is the square root of 9')} \text{ since } 3^2 = 9.$$

$$\sqrt[3]{-8} = -2 \text{ ('-2 is the cubic root of -8')} \text{ since } (-2)^3 = -8.$$

$$\sqrt[4]{625} = 5 \text{ ('5 is the fourth root of 625')} \text{ since } 5^4 = 625.$$

$$\sqrt[5]{243} = 3 \text{ since } 3^5 = 243.$$

$$\sqrt[6]{-64} \notin \mathbb{R} \text{ since there is no real number whose sixth power is negative.}$$

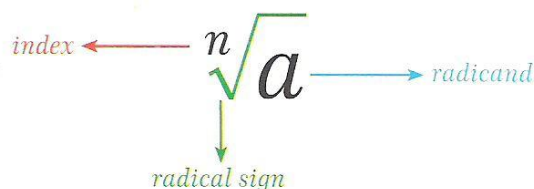
Note

1. An even root is always non-negative, but there is no restriction for an odd root.
2. An even root of a negative number is undefined, but an odd root is defined for any number.

Definition

radical expression, radicand, index

A **radical expression** is an expression of the form $\sqrt[n]{a}$. a is called the **radicand** and n is called the **index**. We do not write the index for square roots: $\sqrt[2]{a} = \sqrt{a}$.



EXAMPLE

40

Evaluate each expression.

a. $\sqrt[3]{64}$

b. $\sqrt[4]{\frac{16}{81}}$

c. $\sqrt[3]{-1}$

d. $\sqrt[3]{-0.001}$

Solution

a. $\sqrt[3]{64} = 4$

b. $\sqrt[4]{\frac{16}{81}} = \frac{2}{3}$

c. $\sqrt[3]{-1} = -1$

d. $\sqrt[3]{-0.001} = -0.1$

Note that when we are evaluating the even root of an expression with variables, the result must always be non-negative. Remember that we guaranteed this result for square roots in the previous section by using the identity $\sqrt{x^2} = |x|$. We can now generalize this for any even root. Note that since there is no restriction for odd roots, we do not need to consider the absolute value.

Definition

Let $n \in \mathbb{N}$, $n \neq 1$. For any real number x ,

1. $\sqrt[n]{x^n} = |x|$ if n is even.
2. $\sqrt[n]{x^n} = x$ if n is odd.

For example, $\sqrt[3]{(-5)^3} = -5$, $\sqrt[4]{(-5)^4} = |-5| = 5$, $\sqrt[5]{(2a-3)^5} = 2a-3$, $\sqrt[4]{(2a-3)^4} = |2a-3|$.

EXAMPLE

41

Simplify the expression $\sqrt{a^2} + \sqrt[3]{b^3} + \sqrt[4]{c^4}$ if $a < 0$, $b < 0$ and $c > 0$.

Solution

$$\sqrt{a^2} + \sqrt[3]{b^3} + \sqrt[4]{c^4} = \underbrace{|a|}_{<0} + b + \underbrace{|c|}_{>0} = -a + b + c$$

2. Properties of Radical Expressions

In the previous section we studied the properties of square roots, i.e. roots with index 2. Now we are ready to generalize these properties for roots with any index.

PROPERTIES OF RADICAL EXPRESSIONS

Let $a \geq 0$, $b \geq 0$ and $k, m, n \in \mathbb{N}$ where $m \neq 1$ and $n \neq 1$. Then

1. $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ ($b \neq 0$)
3. $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
5. $\sqrt[n]{a^m} = \sqrt[nk]{a^{mk}}$

If m and n are odd, these properties also hold for $a < 0$ and $b < 0$.



Proof The proofs of the first three properties are similar to the proofs we gave for the square root. They are left as an exercise for you. We will only prove the last two properties here.

4. Let $\sqrt[n]{\sqrt[m]{a}} = x$. Then by definition, $x^m = \sqrt[n]{a}$.

Again by definition, $(x^m)^n = a$, i.e. $x^{mn} = a$.

Finally, by definition, $x^{mn} = a$ means $x = \sqrt[n]{\sqrt[m]{a}}$. So $x = \sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$.

5. Let $\sqrt[n]{a^m} = x$. Then by definition, $x^n = a^m$.

By the properties of exponents, $(x^n)^k = (a^m)^k$, i.e. $x^{nk} = a^{mk}$.

By the definition, $x^k = a^{mk}$ means $x = \sqrt[k]{a^{mk}}$. So $x = \sqrt[n]{a^m} = \sqrt[nk]{a^{mk}}$.

EXAMPLE 42

Write each expression in its simplest form. All the variables are positive.

- a. $\sqrt[3]{8 \cdot 27 \cdot 125}$ b. $\sqrt[4]{\frac{16}{81}}$ c. $(\sqrt[3]{m^2})^4$ d. $\sqrt[3]{\sqrt{5}}$ e. $\sqrt[16]{a^{12}}$

Solution We can use the properties of radical expressions.

a. By property 1, $\sqrt[3]{8 \cdot 27 \cdot 125} = \sqrt[3]{8} \cdot \sqrt[3]{27} \cdot \sqrt[3]{125} = 2 \cdot 3 \cdot 5 = 30$

b. By property 2, $\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$.

c. By property 3, $(\sqrt[3]{m^2})^4 = \sqrt[3]{(m^2)^4} = \sqrt[3]{m^8}$.

d. By property 4, $\sqrt[3]{\sqrt{5}} = \sqrt[3]{\sqrt[2]{5}} = \sqrt[3 \cdot 2]{5} = \sqrt[6]{5}$. Note that the index of $\sqrt{5}$ is 2 (not 1).

e. By property 5, $\sqrt[16]{a^{12}} = \sqrt[4]{\sqrt[4]{a^{12}}} = \sqrt[4]{a^3}$.

SIMPLIFYING n th ROOTS

$$\sqrt[n]{a^n b} = a \sqrt[n]{b} \quad (a \geq 0, b \geq 0)$$

EXAMPLE 43

Simplify the radicals. All the variables are positive.

- a. $\sqrt[3]{108b^4}$ b. $(\sqrt[3]{a^7})^2$ c. $\sqrt[4]{x^3 \cdot \sqrt[3]{x}}$

Solution a. $\sqrt[3]{108b^4} = \sqrt[3]{2^2 \cdot 3^3 \cdot b^3 \cdot b}$ (write the prime factorization of 108)

$$= \sqrt[3]{(3b)^3 \cdot 2^2 \cdot b}$$

(group the expressions that have a power of 3)

$$= 3b \cdot \sqrt[3]{2^2 \cdot b}$$

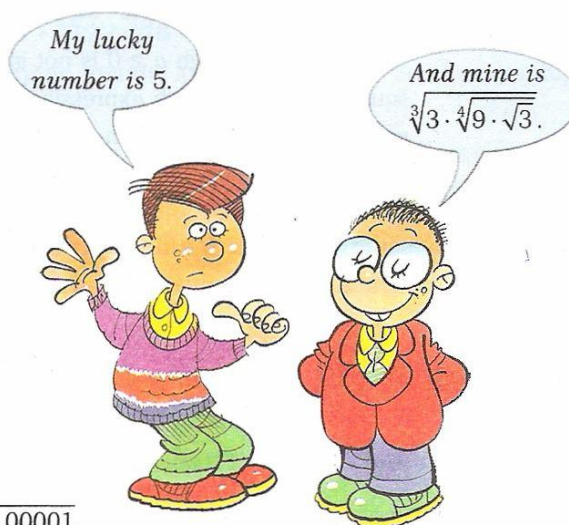
(simplify the radicand by extracting the perfect roots)

$$= 3b \cdot \sqrt[3]{4b}$$



b. $(\sqrt[4]{a^7})^2 = \sqrt[4]{a^{7 \cdot 2}}$ (apply property 3)
 $= \sqrt[4]{a^{14}}$
 $= \sqrt[4]{(a^3)^4 \cdot a^2}$ (group the expressions that have a power of 4)
 $= a^3 \cdot \sqrt[4]{a^2}$ (simplify the radicand by extracting the perfect roots)
 $= a^3 \cdot \sqrt[2]{a^{-1}}$
 $= a^3 \cdot \sqrt[2]{a}$ (apply property 5)
 $= a^3 \cdot \sqrt{a}$

c. $\sqrt[4]{x^3} \cdot \sqrt{x} = \sqrt[4]{(x^3)^5 \cdot x}$ (put x^3 into the second radicand as a perfect root)
 $= \sqrt[4]{x^{16}}$
 $= \sqrt[20]{x^{16}}$ (apply property 4)
 $= \sqrt[5]{x^4}$ (apply property 5)



Check Yourself 9

- Evaluate $\sqrt[4]{81} - \sqrt[3]{-8} + \sqrt[7]{1} + \sqrt[5]{0.00001}$.
- Simplify $\sqrt[4]{16x^{16}y^{12}}$ if $x < 0$ and $y < 0$.
- Simplify the expressions. All the variables are positive.

a. $\sqrt[3]{32a^3b^5}$

b. $(\sqrt[4]{\frac{3a^2}{b^3}})^6$

c. $\sqrt[3]{a^2 \cdot \sqrt[4]{a}}$

Answers

1. 6.1

2. $-2x^4y^3$

3. a. $2ab \cdot \sqrt[3]{4b^2}$

b. $\frac{3a^3}{b^4} \cdot \sqrt{\frac{3}{b}}$

c. $\sqrt[4]{a^3}$

3. Working with Radical Expressions

We can only multiply or divide radical expressions which have the same index. However, we can try to make different indices equal by using property 5 of radical expressions ($\sqrt[n]{a^m} = \sqrt[nk]{a^{mk}}$).

EXAMPLE 44 Simplify the expressions.

a. $\sqrt[4]{a^3} \cdot \sqrt{a}$ b. $\frac{\sqrt{a} \cdot \sqrt[4]{a}}{\sqrt[3]{a}}$ c. $\frac{\sqrt[3]{a^2} \cdot \sqrt[5]{a^2} \cdot \sqrt{a}}{\sqrt[3]{a}}$

Solution a. $\sqrt[4]{a^3} \cdot \sqrt[2]{a^1} = \sqrt[4]{a^3} \cdot \sqrt[4]{a^{2 \cdot 1}} = \sqrt[4]{a^3} \cdot \sqrt[4]{a^2} = \sqrt[4]{a^3 \cdot a^2} = \sqrt[4]{a^5} = a \cdot \sqrt[4]{a}$

b. $\frac{\sqrt{a} \cdot \sqrt[4]{a}}{\sqrt[3]{a}} = \frac{\sqrt[6]{a^{6 \cdot 1}} \cdot \sqrt[12]{a^{3 \cdot 1}}}{\sqrt[12]{a^{4 \cdot 1}}} = \frac{\sqrt[12]{a^6} \cdot \sqrt[12]{a^3}}{\sqrt[12]{a^4}} = \sqrt[12]{\frac{a^6 \cdot a^3}{a^4}} = \sqrt[12]{a^5}$

c. $\frac{\sqrt[3]{a^2} \cdot \sqrt[5]{a^2} \cdot \sqrt{a}}{\sqrt[3]{a}} = \sqrt[3]{\frac{a^2 \cdot \sqrt[5]{a^2} \cdot \sqrt{a}}{a}} = \sqrt[3]{a \cdot \sqrt[5]{a^2} \cdot \sqrt{a}} = \sqrt[3]{\sqrt[5]{\frac{a^5 \cdot a^2}{a^7}} \cdot \sqrt{a}} = \sqrt[3]{\sqrt[5]{(a^7)^2} \cdot a}$
 $= {}^{3 \cdot 5}\sqrt{a^{15}} = {}^{30}\sqrt{a^{15}} = \sqrt{a}$

Notice that although $a \geq 0$ is not given as a condition in this example, the existence of the square root $\sqrt{a}$ in each expression guarantees that $a \geq 0$. Because of this, we can apply our properties without using the absolute value.

We can only easily add or subtract radical expressions which have the same index and the same radicand. If the indices are equal but the radicands are different then we can sometimes make the radicands equal by considering their factors.

EXAMPLE 45 Simplify $\sqrt[4]{a^5} - \sqrt[4]{81a^5} + \sqrt[4]{16a^9}$.

Solution $\sqrt[4]{a^5} - \sqrt[4]{81a^5} + \sqrt[4]{16a^9} = \sqrt[4]{a^4 \cdot a} - \sqrt[4]{(3a)^4 \cdot a} + \sqrt[4]{(2a^2)^4 \cdot a} = a \cdot \sqrt[4]{a} - 3a \cdot \sqrt[4]{a} + 2a^2 \cdot \sqrt[4]{a}$
 $= (a - 3a + 2a^2) \cdot \sqrt[4]{a} = (2a^2 - 2a) \cdot \sqrt[4]{a}$

EXAMPLE 46 Evaluate each expression and write the result in its simplest form.

a. $\sqrt[3]{128} + 4\sqrt[3]{16} + 2\sqrt[3]{54}$ b. $\sqrt{3} \cdot \sqrt[3]{9} \cdot \sqrt[4]{27}$

Solution a. $\sqrt[3]{128} + 4\sqrt[3]{16} + 2\sqrt[3]{54} = \sqrt[3]{2^7} + 4 \cdot \sqrt[3]{2^4} + 2 \cdot \sqrt[3]{2 \cdot 3^3} = 2^2 \cdot \sqrt[3]{2} + 4 \cdot 2 \cdot \sqrt[3]{2} + 2 \cdot 3 \cdot \sqrt[3]{2}$
 $= 4 \cdot \sqrt[3]{2} + 8 \cdot \sqrt[3]{2} + 6 \cdot \sqrt[3]{2} = 18\sqrt[3]{2}$

b. $\sqrt{3} \cdot \sqrt[3]{9} \cdot \sqrt[4]{27} = \sqrt[6]{3^{6 \cdot 1}} \cdot \sqrt[12]{3^{2 \cdot 4}} \cdot \sqrt[12]{3^{3 \cdot 4}} = \sqrt[12]{3^6} \cdot \sqrt[12]{3^8} \cdot \sqrt[12]{3^9} = \sqrt[12]{3^6 \cdot 3^8 \cdot 3^9} = \sqrt[12]{3^{23}} = 3^{12/11}$



EXAMPLE

47

Rationalize the denominator of each fraction.

a. $\frac{5}{\sqrt[3]{3^4}}$

b. $\frac{1}{\sqrt[3]{2}-1}$

c. $\frac{6}{\sqrt[3]{9} + \sqrt[3]{12} + \sqrt[3]{16}}$

Solution

- a. Consider the denominator $\sqrt[3]{3^4}$. If the radicand was not 3^4 but 3^7 , this number would be rational. Therefore, if we multiply both the numerator and the denominator by $\sqrt[3]{3^3}$ we will get a rational denominator:

$$\frac{5}{\sqrt[3]{3^4}} = \frac{5 \cdot \sqrt[3]{3^3}}{\sqrt[3]{3^4} \cdot \sqrt[3]{3^3}} = \frac{5 \cdot \sqrt[3]{3^3}}{\sqrt[3]{3^4 \cdot 3^3}} = \frac{5 \cdot \sqrt[3]{3^3}}{\sqrt[3]{3^7}} = \frac{5 \cdot \sqrt[3]{3^3}}{3}.$$

Algebraic identities:
 $(x-y)^3 = (x-y)(x^2+xy+y^2)$
 $(x+y)^3 = (x+y)(x^2-xy+y^2)$

- b. Multiplying the denominator by its conjugate does not solve our problem because the denominator does not contain square roots but cubic roots. This time we need to use the algebraic identities for the cube of a sum or difference to rationalize the denominator.

$$\frac{1}{\sqrt[3]{2}-1} = \frac{1 \cdot ((\sqrt[3]{2})^2 + \sqrt[3]{2} \cdot 1 + 1^3)}{(\sqrt[3]{2}-1) \cdot ((\sqrt[3]{2})^2 + \sqrt[3]{2} \cdot 1 + 1^3)} = \frac{\sqrt[3]{4} + \sqrt[3]{2} + 1}{(\sqrt[3]{2})^3 - 1^3} = \frac{\sqrt[3]{4} + \sqrt[3]{2} + 1}{2-1} = \sqrt[3]{4} + \sqrt[3]{2} + 1$$

- c. Note that the denominator $\sqrt[3]{9} + \sqrt[3]{12} + \sqrt[3]{16} = (\sqrt[3]{3})^2 + \sqrt[3]{3} \cdot \sqrt[3]{4} + (\sqrt[3]{4})^2$ is similar to the factor $x^2 + xy + y^2$ of the algebraic identity $(x-y)(x^2+xy+y^2)$. So if we multiply it by $\sqrt[3]{3} - \sqrt[3]{4}$ it will be rational.

$$\frac{6}{\sqrt[3]{9} + \sqrt[3]{12} + \sqrt[3]{16}} = \frac{6 \cdot (\sqrt[3]{3} - \sqrt[3]{4})}{(\sqrt[3]{3} - \sqrt[3]{4})(\sqrt[3]{9} + \sqrt[3]{12} + \sqrt[3]{16})} = \frac{6 \cdot (\sqrt[3]{3} - \sqrt[3]{4})}{(\sqrt[3]{3})^3 - (\sqrt[3]{4})^3} = \frac{6 \cdot (\sqrt[3]{3} - \sqrt[3]{4})}{3-4} = \frac{6 \cdot (\sqrt[3]{3} - \sqrt[3]{4})}{-1} = 6\sqrt[3]{4} - 6\sqrt[3]{3}$$

Notice that in **parts b** and **c** of this example we are able to rationalize the denominator quite easily by using an identity. Some denominators are much more difficult to rationalize. We will not consider them here.

Note

To rationalize certain denominators, we can use the following rules.

1. To rationalize $\sqrt[n]{a^m}$, multiply the numerator and denominator by $\sqrt[n]{a^{n-m}}$.
2. To rationalize $\sqrt[n]{a} + \sqrt[n]{b}$, multiply the numerator and denominator by $\sqrt[n]{a^2} - \sqrt[n]{ab} + \sqrt[n]{b^2}$.
3. To rationalize $\sqrt[n]{a} - \sqrt[n]{b}$, multiply the numerator and denominator by $\sqrt[n]{a^2} + \sqrt[n]{ab} + \sqrt[n]{b^2}$.

EXAMPLE

48

Simplify $\sqrt[4]{5 \cdot \sqrt[4]{5 \cdot \sqrt[4]{5 \cdot \dots}}}$.

Solution

Let $\sqrt[4]{5 \cdot \sqrt[4]{5 \cdot \sqrt[4]{5 \cdot \dots}}} = x$. Raising both sides to the fourth power gives us $5 \cdot \sqrt[4]{5 \cdot \sqrt[4]{5 \cdot \dots}} = x^4$.

So $5x = x^4$, which is equivalent to $x(x^3 - 5) = 0$. The solution to this is $x = 0$ or $x = \sqrt[3]{5}$.

Since we are looking for $x = \sqrt[4]{5 \cdot \sqrt[4]{5 \cdot \sqrt[4]{5 \cdot \dots}}} > 0$, the answer is $\sqrt[3]{5}$.



Check Yourself 10

1. Simplify the radical expressions.

a. $\sqrt{a} \cdot \sqrt[3]{a}$

b. $\frac{\sqrt[3]{a^5} \cdot \sqrt[4]{a^5}}{\sqrt{a}}$

c. $\sqrt[4]{16a^9} - \sqrt[4]{81a}$

2. Rationalize the denominator of each fraction.

a. $\frac{6}{\sqrt[4]{3}}$

b. $\frac{3}{1 + \sqrt[3]{4}}$

Answers

1. a. $\sqrt[6]{a^5}$

b. $a^2 \cdot \sqrt[12]{a^5}$

c. $(2a^2 - 3)\sqrt[4]{a}$

2. a. $2 \cdot \sqrt[4]{27}$

b. $\frac{3(1 - \sqrt[3]{4} + \sqrt[3]{16})}{5}$

B. RATIONAL EXPONENTS

1. Understanding Rational Exponents

We are now familiar with integer exponents and the concept of n th root, so we are ready to consider expressions with a rational exponent such as $a^{1/n}$. By the laws for exponents we know that $(a^x)^y = a^{xy}$. So we can write

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a.$$

This means $(a^{1/n})^n = a$, and so by the definition of the n th root, $a^{1/n} = \sqrt[n]{a}$.

Raising both sides of the above equation to the m th power gives us

$$(a^{1/n})^m = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

This leads us to the following definition.

Definition

rational exponent

Let m and n be integers such that $n > 0$. Then $a^{m/n} = \sqrt[n]{a^m}$.

Note that if n is even, a must be non-negative.

For example, $a^{1/3} = \sqrt[3]{a}$, $b^{5/2} = \sqrt{b^5}$, $c^{-2/7} = \frac{1}{\sqrt[7]{c^2}}$.

PUZZLE

Find the mistake.

$$-1 = (-1)^{2/2} = ((-1)^2)^{1/2} = 1^{1/2} = 1$$

EXAMPLE

49

Evaluate the expressions.

a. $9^{1/2}$

b. $(-32)^{3/5}$

c. $25^{-3/2}$

Solution

a. $9^{1/2} = \sqrt{9} = 3$

b. $(-32)^{3/5} = \sqrt[5]{(-32)^3} = \sqrt[5]{(-2)^5}^3 = (\sqrt[5]{(-2)^5})^3 = (-2)^3 = -8$

c. $25^{-3/2} = \frac{1}{25^{3/2}} = \frac{1}{\sqrt{25^3}} = \frac{1}{(\sqrt{25})^3} = \frac{1}{5^3} = \frac{1}{125}$

2. Laws of Rational Exponents

We can easily prove that all five properties of integer exponents are also true for rational exponents, as long as we avoid even roots of negative numbers.

LAWS OF RATIONAL EXPONENTS

Let $a > 0$, $b > 0$ and $m, n \in \mathbb{Q}$. Then

1. $a^m \cdot a^n = a^{m+n}$

2. $\frac{a^m}{a^n} = a^{m-n}$

3. $(a^m)^n = a^{m \cdot n}$

4. $(a \cdot b)^m = a^m \cdot b^m$

5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

The proofs of these laws are similar to the proofs of the laws for integer exponents, and will not be given here.

EXAMPLE

50

Simplify the expressions. ($x > 0$)

a. $x^{1/2} \cdot x^{1/3}$

b. $x^{1/2} \div x^{1/3}$

c. $(x^{1/2})^{1/3}$

d. $x^{1/2} \cdot x^{-1/3} \div x^{-1/6}$

e. $(x^{0.7})^{0.5} \cdot x^{0.15}$

Solution

a. $x^{1/2} \cdot x^{1/3} = x^{1/2 + 1/3} = x^{5/6}$

b. $x^{1/2} \div x^{1/3} = x^{1/2 - 1/3} = x^{1/6}$

c. $(x^{1/2})^{1/3} = x^{(1/2) \cdot (1/3)} = x^{1/6}$

d. $x^{1/2} \cdot x^{-1/3} \div x^{-1/6} = x^{1/2 + (-1/3) - (-1/6)} = x^{1/3}$

e. $(x^{0.7})^{0.5} \cdot x^{0.15} = x^{0.7 \cdot 0.5 + 0.15} = x^{0.5}$

EXAMPLE

51

Evaluate the expressions.

a. $(2^{0.5})^{-0.5} \cdot 0.5^{-1.25}$

b. $(3^{-1/9})^{1.8} \cdot 9^{0.1}$

c. $16^{0.125} \cdot 8^{-5/6} \cdot 4^{2.5}$

Solution

a. $(2^{0.5})^{-0.5} \cdot 0.5^{-1.25} = 2^{-0.25} \cdot (2^{-1})^{-1.25} = 2^{-0.25} \cdot 2^{1.25} = 2^1 = 2$

b. $(3^{-1/9})^{1.8} \cdot 9^{0.10} = 3^{-0.2} \cdot (3^2)^{0.10} = 3^{-0.2 + 0.2} = 3^0 = 1$

c. $16^{0.125} \cdot 8^{-5/6} \cdot 4^{2.5} = (2^4)^{0.125} \cdot (2^3)^{-5/6} \cdot (2^2)^{2.5} = 2^{0.5} \cdot 2^{-2.5} \cdot 2^5 = 2^3 = 8$



Check Yourself 11

1. Evaluate the expressions.

a. $64^{3/2}$

b. $(-81)^{5/4}$

2. Simplify the expressions. All the variables are positive.

a. $(2a^{3/5})(a^{1/3})$

b. $(3x^{-3/4} \cdot x^{2/3})^3$

c. $(x^{1/2} \div x^{1/3})^{1/2}$

Answers

1. a. 512 b. undefined 2. a. $2a^{14/15}$ b. $27x^{-1/4}$ c. $x^{1/12}$

Sometimes it is more convenient to work with rational exponents than with radicals to solve a problem. Let us look at some examples.

EXAMPLE 52

Simplify the expressions.

a. $\sqrt[4]{a^3} \cdot \sqrt{a}$

b. $(\sqrt[3]{a^2})^4 \cdot (\sqrt{a^6})^{1/9}$

c. $\sqrt[9]{\sqrt{a^3} \cdot \sqrt[3]{a^4} \div \sqrt[6]{a^5}} \cdot (a^2 \div \sqrt[3]{a^2})^{1/4}$

Solution

a. $\sqrt[4]{a^3} \cdot \sqrt{a} = a^{3/4} \cdot a^{1/2} = a^{5/4}$ or $\sqrt[4]{a^5}$

b. $(\sqrt[3]{a^2})^4 \cdot (\sqrt{a^6})^{1/9} = (a^{2/3})^4 \cdot (a^{6/2})^{1/9} = a^{(2/3) \cdot 4 + (6/2) \cdot (1/9)} = a^{9/3} = a^3$

c. $\sqrt[9]{\sqrt{a^3} \cdot \sqrt[3]{a^4} \div \sqrt[6]{a^5}} \cdot (a^2 \div \sqrt[3]{a^2})^{1/4} = (a^{3/2} \cdot a^{4/3} \div a^{5/6})^{1/9} \cdot (a^2 \div a^{2/3})^{1/4}$
 $= a^{(3/2 + 4/3 - 5/6) \cdot (1/9)} \cdot a^{(2 - 2/3) \cdot (1/4)}$
 $= a^{5/9}$ or $\sqrt[9]{a^5}$

EXAMPLE 53

Simplify the expressions.

a. $\frac{p^{2/7} - q^{2/7}}{p^{1/7} - q^{1/7}}$

b. $\frac{\sqrt{y} - 16}{3\sqrt[4]{y} + 12}$

c. $\frac{\sqrt{x^3y^2} + \sqrt{x^2y^3}}{\sqrt{x^2y} + \sqrt{xy^2}}$

Solution

a. $\frac{p^{2/7} - q^{2/7}}{p^{1/7} - q^{1/7}} = \frac{(p^{1/7})^2 - (q^{1/7})^2}{p^{1/7} - q^{1/7}} = \frac{(p^{1/7} + q^{1/7})(p^{1/7} - q^{1/7})}{p^{1/7} - q^{1/7}} = p^{1/7} + q^{1/7}$ or $\sqrt[7]{p} + \sqrt[7]{q}$

b. $\frac{\sqrt{y} - 16}{3\sqrt[4]{y} + 12} = \frac{y^{1/2} - 16}{3y^{1/4} + 12} = \frac{(y^{1/4})^2 - 4^2}{3y^{1/4} + 12} = \frac{(y^{1/4} + 4)(y^{1/4} - 4)}{3(y^{1/4} + 4)} = \frac{y^{1/4} - 4}{3}$ or $\frac{\sqrt[4]{y} - 4}{3}$

c. $\frac{\sqrt{x^3y^2} + \sqrt{x^2y^3}}{\sqrt{x^2y} + \sqrt{xy^2}} = \frac{(x^3y^2)^{1/2} + (x^2y^3)^{1/2}}{(x^2y)^{1/2} + (xy^2)^{1/2}} = \frac{x^{3/2}y^{2/2} + x^{2/2}y^{3/2}}{x^{2/2}y^{1/2} + x^{1/2}y^{2/2}} = \frac{xy(x^{1/2} + y^{1/2})}{x^{1/2}y^{1/2}(x^{1/2} + y^{1/2})}$
 $= \frac{xy}{x^{1/2}y^{1/2}} = x^{1-1/2} \cdot y^{1-1/2} = x^{1/2}y^{1/2}$ or $\sqrt{xy}$



EXAMPLE 54 Simplify the expressions.

a. $\left(\frac{a^{1.5} - b^{1.5}}{a^{0.5} - b^{0.5}} + a^{0.5}b^{0.5}\right) \div (a^{0.5} + b^{0.5})$

b. $(b^{0.4} - 4) \div \left(\frac{b^{0.6} + 8}{b^{0.2} + 2} - 2b^{0.2}\right)$

Solution a. $\left(\frac{a^{1.5} - b^{1.5}}{a^{0.5} - b^{0.5}} + a^{0.5}b^{0.5}\right) \div (a^{0.5} + b^{0.5}) = \left(\frac{(a^{0.5})^3 - (b^{0.5})^3}{a^{0.5} - b^{0.5}} + a^{0.5}b^{0.5}\right) \div (a^{0.5} + b^{0.5})$

$$= \left(\frac{(a^{0.5} - b^{0.5})(a + a^{0.5}b^{0.5} + b)}{a^{0.5} - b^{0.5}} + a^{0.5}b^{0.5}\right) \div (a^{0.5} + b^{0.5})$$

$$= (a + a^{0.5}b^{0.5} + b + a^{0.5} \cdot b^{0.5}) \div (a^{0.5} + b^{0.5})$$

$$= \frac{(a^{0.5})^2 + 2a^{0.5}b^{0.5} + (b^{0.5})^2}{a^{0.5} + b^{0.5}}$$

$$= \frac{(a^{0.5} + b^{0.5})^2}{a^{0.5} + b^{0.5}}$$

$$= a^{0.5} + b^{0.5} \text{ or } \sqrt{a} + \sqrt{b}$$

b. $(b^{0.4} - 4) \div \left(\frac{b^{0.6} + 8}{b^{0.2} + 2} - 2b^{0.2}\right) = ((b^{0.2})^2 - 2^2) \div \left(\frac{(b^{0.2})^3 + 2^3}{b^{0.2} + 2} - 2b^{0.2}\right)$

$$= ((b^{0.2})^2 - 2^2) \div \left(\frac{(b^{0.2} + 2)(b^{0.4} - 2b^{0.2} + 4)}{b^{0.2} + 2} - 2b^{0.2}\right)$$

$$= ((b^{0.2})^2 - 2^2) \div (b^{0.4} - 2b^{0.2} + 4 - 2b^{0.2})$$

$$= \frac{(b^{0.2})^2 - 2^2}{b^{0.4} - 4b^{0.2} + 4}$$

$$= \frac{(b^{0.2} + 2)(b^{0.2} - 2)}{(b^{0.2} - 2)^2}$$

$$= \frac{b^{0.2} + 2}{b^{0.2} - 2} \text{ or } \frac{\sqrt[5]{b} + 2}{\sqrt[5]{b} - 2}$$

Check Yourself 12

1. Simplify the expressions.

a. $(b^{-1/2})^{2/3} \cdot \frac{\sqrt[3]{b}}{b^2}$

b. $\sqrt[3]{\sqrt{x^3} \cdot \sqrt{x^4} + \sqrt{x}}$

c. $(\sqrt[3]{a^2})^4 \cdot (\sqrt{a^6})^{1/9}$

2. Simplify $\frac{x-y}{x^{1/3}-y^{1/3}} - (x^{1/3} - y^{1/3})^2$.

Answers

1. a. $\frac{1}{b^2}$

b. x

c. a^3

2. $3\sqrt[3]{xy}$



C. EXPONENTIAL EQUATIONS

An equation which has a variable in an exponent is called an **exponential equation**. When we are solving exponential equations we use the following property: if two exponential expressions have the same index (or base) then they must have the same base (or index).

EXAMPLE 55 Solve the equations.

a. $2^x = 8$

b. $2^{x+2} = 32$

c. $3^{x+4} = \frac{1}{27}$

d. $3^x + 3^{x+2} = 3^3 \cdot 2 \cdot 5$

Solution a. $2^x = 8$

$$2^x = 2^3$$

$$x = 3$$

b. $2^{x+2} = 32$

$$2^{x+2} = 2^5$$

$$x + 2 = 5$$

$$x = 3$$

c. $3^{x+4} = 27^{-1}$

$$3^{x+4} = (3^3)^{-1}$$

$$3^{x+4} = 3^{-3}$$

$$x + 4 = -3$$

$$x = -7$$

d. $3^x + 3^{x+2} = 3^3 \cdot 2 \cdot 5$

$$3^x + 3^x \cdot 3^2 = 3^3 \cdot 10$$

$$3^x(1 + 9) = 3^3 \cdot 10$$

$$3^x \cdot 10 = 3^3 \cdot 10$$

$$3^x = 3^3$$

$$x = 3$$

EXAMPLE 56 Find x if $\frac{1}{25^x} = 0.000064$.

Solution $\frac{1}{25^x} = 0.000064$. Let us try to get the same base on each side of the equation.

$$\frac{1}{(5^2)^x} = 64 \cdot 10^{-6} = 2^6 \cdot 10^{-6} = \frac{2^6}{10^6}, \text{ so } \frac{1}{5^{2x}} = \frac{2^6}{10^6}.$$

This gives $(\frac{1}{5})^{2x} = (\frac{2}{10})^6$, i.e. $(\frac{1}{5})^{2x} = (\frac{1}{5})^6$ and so $2x = 6$, $x = 3$.

EXAMPLE 57 Solve $2^x + 2^{x+1} + 2^{x+2} = 56$.

Solution $2^x + 2^{x+1} + 2^{x+2} = 56$ means $2^x + (2^x \cdot 2) + (2^x \cdot 2^2) = 56$

$$2^x(1 + 2 + 4) = 56$$

$$7 \cdot 2^x = 56$$

$$2^x = 8, x = 3.$$

Check Yourself 13

Solve the equations.

1. $4^x = 2$

2. $2^{-x+1} = 8$

3. $5^x = 1$

4. $(0.25)^x = 4^2$

Answers

1. 0.5

2. -2

3. 0

4. -2



EXERCISES 2.3

A. Radical Expressions

1. Evaluate the expressions.

- a. $\sqrt[4]{-16}$ b. $-\sqrt[4]{16}$ c. $\sqrt[4]{16}$
 d. $\sqrt[3]{\frac{27}{8}}$ e. $\sqrt{\frac{25}{4}}$ f. $\sqrt{25^{-3}}$
 g. $\sqrt[3]{27^{-2}}$ h. $\sqrt[4]{0.0625}$ i. $\sqrt[7]{-128}$

2. Write each expression in its simplest form. All the variables represent positive real numbers.

- a. $\sqrt{4x^4y^{12}}$ b. $\sqrt[3]{25a^6y^{12}}$
 c. $\sqrt[4]{64x^4y^{20}}$ d. $\sqrt[5]{32x^{15}y^{20}}$
 e. $\sqrt[5]{4xy}$ f. $\sqrt[3]{\sqrt[3]{x^2y^3}}$
 g. $\sqrt[4]{x^{16}}$ h. $\sqrt[3]{9a^2} \cdot \sqrt[3]{9a}$
 i. $\sqrt[4]{x} \cdot \sqrt[3]{y^5}$ j. $\frac{\sqrt[12]{243x^6y^{12}}}{xy}$
 k. $\sqrt[16]{3^{12}(x+y)^{12}}$ l. $\sqrt[4]{\sqrt[3]{(x-a)^{20}(a-x)^{28}}}$

3. Write each expression in its simplest form. All the variables represent positive real numbers.

- a. $\sqrt[3]{x^2} \cdot \sqrt[5]{x^5}$
 b. $\frac{\sqrt{x^5} \cdot \sqrt[4]{x^3}}{x \cdot \sqrt[3]{x^2}}$
 c. $(x^2 \cdot \sqrt{xy} \cdot \sqrt[3]{y^2})^2 \cdot \sqrt[4]{x^3}$
 d. $\sqrt[4]{5^8x} - \sqrt[4]{3^4x^5} + \sqrt[4]{25^2x^9}$
 e. $\frac{\sqrt[4]{2x} \cdot \sqrt[4]{x^3} \cdot \sqrt[4]{4x^2}}{\sqrt{2xy} \cdot \sqrt{2x} \cdot \sqrt{2y}}$

4. Write each expression in its simplest form.

- a. $\sqrt[3]{16} - \sqrt[3]{54} + \sqrt[3]{128}$
 b. $\sqrt[4]{32} + \sqrt[4]{162} - \sqrt[4]{512}$
 c. $\sqrt[4]{162} - \sqrt[3]{54} + \sqrt[4]{32} + \sqrt[3]{16}$

5. Rationalize the denominator of each fraction.

- a. $\frac{1}{\sqrt[3]{5}}$ b. $\frac{x}{\sqrt[4]{x}}$ c. $\frac{\sqrt[5]{2}}{\sqrt[3]{2}}$
 d. $\frac{4}{\sqrt[3]{2}-1}$ e. $\frac{1}{\sqrt[3]{x}+1}$ f. $\frac{\sqrt[12]{x^{11}}-1}{\sqrt{\sqrt[3]{x}}}$
 g. $\frac{1}{\sqrt[3]{3}-\sqrt[3]{2}}$ h. $\frac{1}{\sqrt[4]{2}-1}$ i. $\frac{1}{\sqrt[3]{16}-\sqrt[3]{12}+\sqrt[3]{9}}$
 j. $\frac{4}{(\sqrt[16]{3}+1)(\sqrt[8]{3}+1)(\sqrt[4]{3}+1)(\sqrt{3}+1)}$

6. Evaluate each expression.

- a. $\sqrt{56} + \sqrt{56} + \sqrt{56} + \dots$
 b. $\sqrt[6]{32} \cdot \sqrt[6]{32} \cdot \sqrt[6]{32} \cdot \dots$
 c. $\sqrt[4]{32} + \sqrt[4]{32} + \sqrt[4]{32} + \dots$

7. Prove each equality.

- a. $\sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \dots = \sqrt[n-1]{a}$
 b. $\sqrt[n]{a + \sqrt[n]{a + \sqrt[n]{a} + \dots}} = \sqrt[n+1]{a}$
 c. $\sqrt[n]{a} \cdot \sqrt[m]{a} \cdot \sqrt[n]{a} \cdot \sqrt[m]{a} \cdot \dots = \sqrt[mn-1]{a^{m+1}}$
 d. $\sqrt{a + \sqrt{a + \sqrt{a} + \dots}} = \frac{\sqrt{4a+1}+1}{2}$

B. Rational Exponents

8. Evaluate each expression.

- | | | |
|---------------|-----------------|-----------------|
| a. $9^{1/2}$ | b. $-9^{1/2}$ | c. $(-9)^{1/2}$ |
| d. $-8^{1/3}$ | e. $(-8)^{1/3}$ | f. $0^{1/16}$ |
| g. $16^{3/4}$ | h. $32^{4/5}$ | |

9. For each statement, give a possible value for a .

- | | |
|-------------------------|-------------------------|
| a. $(a^2)^{1/2} \neq a$ | b. $(a^2)^{1/2} = a$ |
| c. $(a^2)^2 = a$ | d. $(a^{1/2})^2 \neq a$ |
| e. $(a^{1/2})^2 = a$ | |

10. Evaluate the expressions.

- | | |
|---|--|
| a. $(-8)^{7/3} \cdot 9^{1/2}$ | b. $64^{1/3} + 3^{1/2} \cdot 9 \cdot 3^{-5/2}$ |
| c. $4^{1.5} - (\frac{1}{9})^{-0.5} + (\frac{1}{64})^{-2/3} + (\frac{5}{4})^{3.5} \cdot 0.8^{3.5}$ | |

11. Simplify the expressions. All the variables represent positive real numbers.

- | | |
|--|---|
| a. $(2x^{1/2}) \cdot (3x^{1/3})$ | b. $x^{2/5} \cdot 25^{1/2} \cdot x^{3/5}$ |
| c. $(\frac{4x^{5/2}}{x^{2/5}})^{1/2}$ | d. $(x^{1/2} - y^{1/2})(x^{1/2} + y^{1/2})$ |
| e. $\frac{36^{1/2} \cdot x^{1/4}}{64^{1/2} \cdot x^{3/5}}$ | f. $(2x^{-1/2} \cdot y^{2/5})^3$ |

12. Write each expression as a radical expression. All the variables represent positive real numbers.

- | | | |
|----------------------|----------------------|---------------------|
| a. $a^{3/5}$ | b. $x^{-1/3}$ | c. $a^{-4/5}$ |
| d. $x^{-1/2}y^{1/2}$ | e. $(x^{1/2})^{1/3}$ | f. $x^{-2/7}$ |
| g. $y^{-5/3}$ | h. $5a^{3/2}$ | i. $(x^2y^3)^{1/2}$ |
| j. $(4ab^2)^{5/7}$ | k. $(7x^3y)^{-1/3}$ | l. $(a+b)^{3/2}$ |

13. Write each expression as an expression with a rational exponent. All the variables represent positive real numbers.

- | | |
|----------------------------|------------------------------------|
| a. $\sqrt[3]{x}$ | b. $\sqrt[4]{a}$ |
| c. $\sqrt{x^2 + y^2}$ | d. $-\sqrt[5]{(3a)^2}$ |
| e. $\sqrt[3]{(x-y)^{-1}}$ | f. $\sqrt[6]{x^{12}y^{12}}$ |
| g. $\sqrt{a} + \sqrt{b}$ | h. $3\sqrt[4]{a^3}$ |
| i. $5x\sqrt[5]{y^2}$ | j. $\sqrt[7]{(x^2y^3)^3}$ |
| k. $\sqrt[9]{(x^{-3}y)^2}$ | l. $\sqrt[3]{a} + \sqrt[5]{x+y^2}$ |

14. Simplify each expression by converting the radicals into rational powers.

- | |
|--|
| a. $\frac{\sqrt[4]{a^2} \cdot \sqrt[3]{a}}{\sqrt{a}}$ |
| b. $\frac{c^{4/5} \cdot \sqrt[3]{a^{-2}}}{\sqrt[5]{c^{-1}}} \cdot (8a^{-2}c)^{-1/3}$ |

15. Simplify each expression.

- | |
|---|
| a. $(x^{1/2} + 2y^{1/2})(x^{1/2} - 3y^{1/2})$ |
| b. $(3a^{1/2} - b^{1/2})(a^{1/2} - 2b^{1/2})$ |
| c. $2x^{1/3}(3x^{2/3} - x^{8/3})$ |
| d. $3a^{4/5}(2a^{1/5} - a^{6/5})$ |
| e. $(3x^{1/2} - 2y^{1/2})(3x^{1/2} + 2y^{1/2})$ |
| f. $(3x^{1/2} + y^{1/2})^2$ |

16. Simplify each expression.

- | |
|---|
| a. $\frac{a^{2/5} - b^{2/5}}{a^{1/5} - b^{1/5}}$ |
| b. $\frac{a^{3/4} + b^{3/4}}{a^{1/2} - a^{1/4}b^{1/4} + b^{1/2}}$ |
| c. $\frac{x+y^{1/2}}{x^{1/3} + y^{1/6}} - \frac{x^{4/3} - y^{2/3}}{x^{2/3} + y^{1/3}} + x^{1/3}y^{1/6}$ |
| d. $\frac{c-27}{c^{2/3}-9} - \frac{3c^{1/3}+18}{c^{1/3}+3}$ |



C. Exponential Equations

17. Solve the equations.

- a. $3^{2x} = 81$
- b. $7^x = 7^{2-x}$
- c. $25^x = 5^{3-x}$
- d. $3^{x^2-5x+8} = 9$

18. $(\frac{2}{3})^x = \sqrt[4]{1.5}$ is given. Find x .

19. Find the real number y which satisfies $x = 2^{2y+4}$ and $x^3 = 2^{10y+4}$.

20. Find the real numbers x and y which satisfy $0.0064^y \cdot 1000^x = 4^5$.

21. Find the real number x which satisfies $(-4x + 1)^{2n} = (x^2 + 2x + 1)^n, n \in \mathbb{N}$.

22. Find the possible values of a which satisfy $2^{a+1} + 4^a = 80$.

23. Find the real number x which satisfies $\frac{x^{m-15} + x^{m-16} + x^{m-17}}{x^{m-18} + x^{m-17} + x^{m-16}} = 128$.

Mixed Problems

24. Evaluate

$$[a^{-3/2}b(ab^{-2})^{-1/2}(a^{-1})^{-2/3}]^3 \text{ if } a = \frac{\sqrt{2}}{2} \text{ and } b = \frac{1}{\sqrt[3]{2}}.$$

25. Evaluate $\frac{\sqrt[4]{8} + \sqrt{\sqrt{2}-1} - \sqrt[4]{8} - \sqrt{\sqrt{2}-1}}{\sqrt[4]{8} - \sqrt{\sqrt{2}+1}}$.

26. Order the numbers $\sqrt[16]{64}, \sqrt[10]{7\sqrt[4]{7}}$ and $\sqrt[4]{2\sqrt{\frac{5}{4}}}$.

27. Find the sum of all values of x which satisfy $(x-3)^{x^2-6x+5} = 1$.

28. Find the possible values of x which satisfy $\sqrt[3]{25} \cdot \sqrt[3]{25} \cdot \sqrt[3]{25} \dots = 25$.

29. Solve the equation $(7 \cdot \sqrt[3]{5^{x-6}}) + 90 = \sqrt[3]{5^x}$.

30. Solve the equation $\sqrt[3]{5^{x-3}} + \frac{10}{\sqrt[3]{5^{x-3}}} - 7 = 0, x \in \mathbb{Z}$.

31. Solve the equation $\sqrt[3]{2} \cdot 2^x = 4$.

32. Solve the equation $\sqrt{3-\sqrt{5}} + \sqrt{3+\sqrt{5}} = 10^{\sqrt{x+2-2\sqrt{x+1}}}$.

CHAPTER SUMMARY

1. Integer Exponents

- For any real number a and any positive integer n ,

$$1. a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$$

$$2. a^0 = 1 \quad (0^0 \text{ is undefined})$$

$$3. a^{-n} = \frac{1}{a^n} \quad \text{where } a \neq 0.$$

- When moving expressions from the numerator to the denominator (or from the denominator to the numerator), simply change the sign of the exponent.
- All powers of a positive real number are positive. All even powers of a negative real number are positive, and all odd powers of a negative real number are negative.
- $(-a)^n$ and $-(a^n)$ are not the same. In $(-a)^n$ the base is $-a$, but in $-(a^n)$ the base is a .

Laws of Integer Exponents

For $a, b \in \mathbb{R}$, $b \neq 0$ and $m, n \in \mathbb{Z}$,

$$1. a^m \cdot a^n = a^{m+n}$$

$$2. \frac{a^m}{a^n} = a^{m-n}$$

$$3. (a^m)^n = a^{m \cdot n}$$

$$4. (a \cdot b)^n = a^n \cdot b^n$$

$$5. \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

- As a consequence of the power rules for integer exponents, the following properties hold.

$$1. (a^m \cdot b^n)^p = a^{mp} \cdot b^{np}$$

$$2. \left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}$$

$$3. \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

- A number is expressed in **scientific notation** when it is in the form $a \cdot 10^n$, where $1 < |a| < 10$ and n is an integer. If a number is written in normal notation, we say it is in **standard notation**.

2. Square Roots

- Let a be a non-negative real number. If $b^2 = a$ then b is a **square root** of a . The non-negative square root of a is called the **principal square root** of a .
- The symbol $\sqrt{\quad}$, called a **radical sign**, is used to represent the principal square root of a number. The number under the radical sign is called a **radicand**. An expression which comprises a radical sign and a radicand is called a **radical expression**.

$$\bullet \text{ For any real number } x, \sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Properties of Square Roots

For any $a \geq 0$, $b \geq 0$ and $n \in \mathbb{Z}$,

$$1. \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

$$2. \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (b \neq 0)$$

$$3. (\sqrt{a})^n = \sqrt{a^n}$$

- For all $a \geq 0$, $b \geq 0$, $\sqrt{a^2 b} = a \sqrt{b}$.
- For all $a, b \in \mathbb{R}^+$, $a < b$ if and only if $\sqrt{a} < \sqrt{b}$.
- Two square root expressions can only be added or subtracted if they have the same radicand.
- An expression with exactly two terms is called a **binomial expression**. Two binomial expressions whose first terms are equal and last terms are opposite are called **conjugates**. For example, $x + y$ and $x - y$ are conjugates.
- Changing the denominator of a fraction from an irrational number to a rational number is called **rationalizing the denominator** of the fraction. To rationalize the denominator, we multiply the numerator and the denominator of the fraction by a suitable number to make the denominator a rational number.
- $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$
- $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$

3. Radical Expressions and Rational Exponents

- Let a and b be real numbers and let n be a natural number different from 1 such that $b^n = a$. Then b is called the **principal n th root** of a , denoted by $b = \sqrt[n]{a}$.
- An even root is always non-negative, but there is no restriction for an odd root. An even root of a negative number is undefined, but an odd root is defined for any number.
- In the expression $\sqrt[n]{a}$, a is called the **radicand** and n is called the **index**. We do not write the index for square roots: $\sqrt{a} = \sqrt[2]{a}$.

Properties of Radical Expressions:

Let $a \geq 0$, $b \geq 0$ and $k, m, n \in \mathbb{N}$ where $m \neq 1$ and $n \neq 1$.

$$1. \sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b} \qquad 2. \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (b \neq 0)$$

$$3. (\sqrt[n]{a})^m = \sqrt[n]{a^m} \qquad 4. \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$5. \sqrt[n]{a^m} = \sqrt[nk]{a^{mk}}$$

If m and n are odd, these properties also hold for $a < 0$ and $b < 0$.

- For all $a \geq 0$ and $b \geq 0$, $\sqrt[n]{a^n b} = a \sqrt[n]{b}$.
- We can only multiply or divide radical expressions which have the same index. We can only easily add or subtract radical expressions which have the same index and the same radicand.
- To rationalize a denominator with an n^{th} root, multiply the numerator and the denominator by
 - $\sqrt[n]{a^{n-m}}$ if the denominator is $\sqrt[n]{a^m}$.
 - $\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}$ if the denominator is $\sqrt[3]{a} + \sqrt[3]{b}$.
 - $\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$ if the denominator is $\sqrt[3]{a} - \sqrt[3]{b}$.
- Let m and n be integers such that $n > 0$. Then $a^{m/n} = \sqrt[n]{a^m}$. If n is even then a must be non-negative.

Laws of Rational Exponents

Let $a > 0$, $b > 0$ and $m, n \in \mathbb{Q}$.

$$1. a^m \cdot a^n = a^{m+n} \qquad 2. \frac{a^m}{a^n} = a^{m-n}$$

$$3. (a^m)^n = a^{m \cdot n} \qquad 4. (a \cdot b)^n = a^n \cdot b^n$$

$$5. \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

- An equation which has a variable in an exponent is called an **exponential equation**. If two exponential expressions have the same index (or base) then they must have the same base (or index).

Concept Check

- Which powers of a negative real number are positive?
- Is the negative power of a real positive number negative?
- Why is scientific notation useful?
- What is the difference between the square root and the principal square root of a number?
- What is a radicand? Can it be negative? Explain your answer.
- Explain the difference between these two statements.
 - If $x^2 = 4$ then $x = \pm 2$.
 - $\sqrt{4} = 2$
- Is it always true that $\sqrt{x^2} = x$? Explain your answer.
- Is the inequality $\sqrt{\frac{1}{2}} > \frac{1}{2}$ true?
- How can we add two square root expressions if they have different radicands?
- Explain the meaning of conjugate.
- What is the difference between an expression with a rational exponent and radical expression?
- What is the n th root of zero?
- Can an even root be negative? What about an odd root? Explain your answer.
- How can we move a product which is in front of a radical sign to the radicand?
- How can we multiply two radical expressions with different indices?
- What does rationalizing the denominator mean? Why is it useful?
- State the laws of exponents. Is it true that they hold for any real base? Explain your answer.
- Can we find an irrational power of a real number?
- What is an exponential equation? How can we solve such equations?



CHAPTER REVIEW TEST 2A

1. Evaluate $4^3 \cdot 8^2 \cdot 16^3$.

- A) 2^{12} B) 2^{24} C) 4^{24} D) 8^{12} E) 16^8

2. Solve $9^{x+2} = 243$ for x .

- A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) 1 D) 2 E) 3

3. Simplify $2^k \cdot 7^k \cdot (-2)^{-k}$ if k is an odd integer.

- A) 14^k B) 27^k C) 7^{-k} D) 7^k E) $(-7)^k$

4. $a = (8^4)^7$, $b = (4^5)^8$ and $c = 2^{(3^4)}$ are given.
Which one of the following is correct?

- A) $a < b < c$ B) $c < b < a$
C) $c < a < b$ D) $b < c < a$
E) $b < a < c$

5. Evaluate $81^{0.25} + 49^{0.5} + 32^{0.2}$.

- A) 10 B) 12 C) 14 D) 16 E) 20

6. Simplify $2^{3a-b} \cdot 2^{2b-a} + 2^{a-b} \cdot 2^{a+2b}$.

- A) 2^{2b+a+1} B) 2^{b-2a} C) 2^{2a+b+1}
D) 2^{2a-b} E) 2^{2a+2}

7. Simplify $\frac{(2^{x+1})^3}{(2^{x+2})^2}$.

- A) 2^{x-1} B) 2^{x+1} C) 2^{2x+1}
D) 2^{x-2} E) 2^{2x+2}

8. Simplify $\frac{2^{3t} + 2^{3t} + 2^{3t} + 2^{3t} + 2^{3t} + 2^{3t} + 2^{3t} + 2^{3t}}{8^{t+1}}$.

- A) 0 B) 1 C) 2 D) 2^t E) 2^{t+1}

9. Evaluate $100^{-3} \cdot 5^8 \cdot 4^5$.

- A) 200 B) 400 C) 50 D) 100 E) 250

10. Evaluate $\sqrt{4 + \sqrt{23 + \sqrt{4}}}$.

- A) 3 B) 4 C) 5 D) 6 E) 8

11. Evaluate $\sqrt{0.64} + \sqrt{0.09} - 2\sqrt{2.25}$.

- A) -2.4 B) -2.2 C) -1.9 D) -1.1 E) -0.9

12. Simplify $\sqrt{18k^5} \cdot \sqrt{72k^3}$.

- A) $6k^4$ B) $36k^4$ C) $\sqrt{6}k$ D) $\sqrt{k^3}$ E) $36k$

13. Evaluate $\sqrt{\frac{3}{8} + \sqrt{2 + \frac{1}{4}}}$.

- A) $\frac{3}{2}$ B) $\frac{\sqrt{3}}{2}$ C) $\sqrt{18}$
D) $\sqrt{\frac{5}{4}}$ E) $\sqrt{\frac{15}{8}}$

14. Evaluate $5 \cdot \sqrt[3]{0.008}$.

- A) $\sqrt[3]{0.04}$ B) $\sqrt[3]{0.2}$ C) 0.01 D) 0.1 E) 1

15. Evaluate $\frac{\sqrt{2} + 2\sqrt{6}}{2\sqrt{2} - \sqrt{6}}$.

- A) $\frac{12\sqrt{6}}{5}$ B) 12 C) $2\sqrt{6} - 2$
D) $8 + 5\sqrt{3}$ E) $2 + 2\sqrt{6}$

16. Order $5\sqrt{2}$, $4\sqrt{3}$ and 7.

- A) $4\sqrt{3} < 5\sqrt{2} < 7$ B) $5\sqrt{2} < 7 < 4\sqrt{3}$
C) $7 < 4\sqrt{3} < 5\sqrt{2}$ D) $4\sqrt{3} < 7 < 5\sqrt{2}$
E) $7 < 5\sqrt{2} < 4\sqrt{3}$

CHAPTER REVIEW TEST 2B

1. Solve $9^{x+2} = 240 + 9^x$.

- A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) 1 D) $\frac{3}{2}$ E) $\frac{5}{2}$

2. Simplify $\frac{(-\frac{1}{3})^{-3} \cdot 27^{-2}}{(-9)^{-3}}$.

- A) $-\frac{1}{27}$ B) -27 C) -9 D) 27 E) 9

3. Given $\begin{cases} 3^x + 4^y = 793 \\ 3^x - 4^y = 665 \end{cases}$, find 5^{x-y} .

- A) $625\sqrt{5}$ B) 125 C) 625
D) 3125 E) $3125\sqrt{5}$

4. Given $25^{3x+1} = a$, find 125^{2x+1} in terms of a .

- A) a B) a^2 C) a^5 D) $4a$ E) $5a$

5. Simplify $\frac{-2^{2n+3} + (-2)^{2n+3}}{(-2)^{2n} - (-2^{2n})}$.

- A) -8 B) -2^{2n} C) -2 D) 2 E) 2^{2n}

6. Evaluate $\frac{2^3 + 2^7 + 2^9}{2^2 + 2^6 + 2^8}$.

- A) 2^4 B) 2^3 C) 2^2 D) 2 E) 1

7. Evaluate $\sqrt{4 + \sqrt{12}} + \sqrt{7 + 4\sqrt{3}}$.

- A) $\sqrt{3}$ B) $2\sqrt{3}$ C) $1 + \sqrt{3}$
D) $3 + 2\sqrt{3}$ E) $4 + 4\sqrt{3}$

8. Evaluate $\sqrt{(-4)^2} - \sqrt[3]{(-3)^3} + \sqrt{25}$.

- A) -10 B) -2 C) 10 D) 12 E) 14



9. Evaluate $243^{0.2} + 64^{0.666\dots} - 125^{0.333\dots} - (-11)^0$.

- A) -10 B) -6 C) 0 D) 10 E) 13

10. Evaluate $\frac{1}{\sqrt{3}-\sqrt{2}} - \frac{2}{\sqrt{2}}$.

- A) $\sqrt{2}$ B) $\sqrt{3}$ C) $\frac{3\sqrt{2}}{2}$ D) $\frac{2\sqrt{3}}{3}$ E) 0

11. Evaluate $\frac{20\sqrt{45} - 15\sqrt{32} + 12\sqrt{75}}{15\sqrt{20} + 10\sqrt{27} - 6\sqrt{50}}$.

- A) 1 B) 2 C) $\sqrt{2}$ D) $\sqrt{3}$ E) $\sqrt{5}$

12. Evaluate $\sqrt{2+\sqrt{2}} \cdot \sqrt{2-\sqrt{2}}$.

- A) 2 B) 3 C) $\sqrt{2}$ D) $\sqrt{3}$ E) $\sqrt{6}$

13. $a = \sqrt{5} + \sqrt{3}$, $b = \sqrt{6} + \sqrt{2}$ and $c = \sqrt{7} + 1$ are given. Order a , b and c .

- A) $a > b > c$ B) $b > a > c$ C) $a > c > b$
D) $b > c > a$ E) $c > b > a$

14. Evaluate $\frac{6}{\sqrt[3]{2+\sqrt{2}}}$.

- A) $\sqrt[3]{2+\sqrt{2}}$ B) $\sqrt[3]{2}$ C) $3 \cdot \sqrt[3]{8-4\sqrt{2}}$
D) 12 E) $3\sqrt[3]{2-\sqrt{2}} \cdot \sqrt{2+\sqrt{2}}$

15. Evaluate $\frac{0.01^{-4} \cdot 0.1^{-5}}{0.0001^2}$.

- A) 10^{18} B) 10^{19} C) 10^{20} D) 10^{21} E) 10^{22}

16. Find the sum of the roots of the equation

$$\sqrt{x^2 - 4x + 4} = 3.$$

- A) 0 B) 2 C) 3 D) 4 E) 5

CHAPTER REVIEW TEST 2C

1. Find x if $\sqrt{x} \cdot \sqrt{x} \cdot \sqrt{x} = \sqrt[4]{4^5}$.

- A) $4\sqrt[3]{64}$ B) $\sqrt[3]{64}$ C) $\sqrt[10]{4^7}$
D) 4 E) 64

2. Evaluate $(\sqrt{7} + \sqrt{5})(\sqrt[4]{7} + \sqrt[4]{5})(\sqrt[4]{7} - \sqrt[4]{5})$.

- A) 12 B) $2\sqrt{7} - 2\sqrt{5}$ C) $2\sqrt{7} + 2\sqrt{5}$
D) 2 E) 1

3. Simplify $(0.5a^{0.25} + a^{0.75})^2 - a^{0.5}(0.25 + a^{0.5})$.

- A) $\sqrt{a^6}$ B) $\sqrt{a^5}$ C) $\sqrt{a^4}$ D) $\sqrt{a^3}$ E) $\sqrt{a}$

4. Find the sum of the roots of the equation

$$3^{2x+2} - 3^{x+3} - 3^x + 3 = 0.$$

- A) $-\frac{5}{2}$ B) -2 C) $-\frac{3}{2}$ D) -1 E) $-\frac{1}{2}$

5. Solve $7^y \cdot 49^{2-y} = 343\sqrt{7}$.

- A) $-\frac{1}{2}$ B) 2 C) $\frac{1}{2}$ D) $\sqrt{2}$ E) -2

6. Simplify $\frac{2a}{\sqrt{a+1} - \sqrt{1-a}} - \sqrt{1-a}$.

- A) $\frac{1-a}{\sqrt{1+a^2}}$ B) $\sqrt{1+a}$ C) $\sqrt{a^2-1}$
D) $\sqrt{1-a^2}$ E) $\frac{1-\sqrt{1+a^2}}{a}$

7. Given $A = \frac{\sqrt{5}+1}{\sqrt{2}+1}$, find $\frac{\sqrt{5}-1}{\sqrt{2}-1}$ in terms of A .

- A) $\frac{2}{A}$ B) $\frac{3}{A}$ C) $\frac{4}{A}$ D) $\frac{5}{A}$ E) $\frac{6}{A}$

8. Evaluate $\frac{\sqrt{2} + \sqrt{5}}{\sqrt{2} + \sqrt{5} + \sqrt{6} + \sqrt{15}}$.

- A) $\frac{1+\sqrt{3}}{2}$ B) $-\frac{1+\sqrt{3}}{2}$ C) $\frac{\sqrt{2}+1}{3}$
D) $\frac{1-\sqrt{2}}{3}$ E) $\frac{\sqrt{3}-1}{2}$

9. Find the smallest integer value of x which satisfies

$$\sqrt{x} - \sqrt{x-1} < \frac{1}{14}.$$

A) 48 B) 49 C) 50 D) 51 E) 52

10. Given $\begin{cases} a = 9^x + 5 \\ b = 3 - 3^x \end{cases}$, find a in terms of b .

A) $3 - b$ B) $b^2 - 3b$ C) $b^2 + 4$
D) $b^2 - 6b + 7$ E) $b^2 - 6b + 14$

11. Evaluate $\sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}} + \sqrt{6 - \sqrt{6 - \sqrt{6 - \dots}}}$.

A) 2 B) 3 C) 5 D) 6 E) 12

12. $a = \sqrt{x-1}$ and $b = x^2 + x + 1$ are given.

Find $a^3 b^{1/2}$.

A) $x - 1$ B) $(x-1)\sqrt{x^3-1}$ C) $\sqrt{x+1}$
D) $(x^3-1)\sqrt{x-1}$ E) $x^3 - 1$

13. Evaluate $\frac{\sqrt{42 + \sqrt{42 + \dots}}}{\sqrt[3]{16 + \sqrt[3]{16 + \dots}}}$.

A) 3 B) $\frac{7}{2}$ C) 4 D) $\frac{9}{2}$ E) 5

14. Order the numbers $a = \sqrt{3}$, $b = \sqrt[3]{4}$ and $c = \sqrt[4]{6}$.

A) $a < b < c$ B) $c < b < a$
C) $c < a < b$ D) $b < c < a$
E) $b < a < c$

15. Given $2^{x-y} + 2^{y-x} = 11$, find $4^{x-y} + 4^{y-x}$.

A) 119 B) 121 C) 123 D) 22 E) 44

16. Given $\sqrt[3]{x+5} = \sqrt{x+1}$, find the sum of all possible values of x .

A) 3 B) 7 C) 12 D) 16 E) 20

ANSWERS TO EXERCISES

EXERCISES 1.1

1. a. no b. no c. yes d. yes 2. $\{3, 4, 5\}$ 3. $\{6, 8\}$ 4. $\{3, 5\}$ 5. 1278 6. 210 7. 4 8. 7 9. $m = 6, n = 5$ 10. 72
 11. 4 12. 6795 13. 16 14. $BC = 18A^2$ 15. use $2n + 1$ for an odd number and $2n$ for an even number 16. a. 216
 b. 128 c. 100000 d. 82 e. 243 17. a. 5^7 b. a^5 c. 6^2 d. 2^3 e. a^3 f. b^6 18. a. a^6 b. 2^{6m} c. x^{11} d. m^{12} e. n^{20} f. 3^{60} g. x^{26}
 h. a^{33} i. 3^{28a} 19. a. $2^3 \cdot 5^5$ b. $2^6 \cdot 5^4 \cdot 3^4$ c. $5^8 \cdot 3^{16} \cdot 2^{12}$ d. $3^4 \cdot 2$ 20. a. 3^{38} b. $2^{46} \cdot 3^{14}$ 21. a. m^2 b. $27m^3$ c. $81^5 \cdot m^8$
 d. $243m^{15}$ 22. a. a^8b^7 b. $x^{10}y^{10}$ c. a^8b^9 d. $8m^{10}n^{11}$ e. $2^{18} \cdot a^{17}$ f. $x^{26}y^{17}$ g. $2^{11} \cdot 3^7 \cdot a^{17}b^{17}c^{20}$ h. $3^5 \cdot 5^8 \cdot x^{17}y^7z^4$
 23. a. 7 b. 19 c. 24 d. 2 24. 125 25. 3 26. a. 1 b. 45 c. 84 27. 5 28. a. 1 b. 7 c. 25 d. 43
 29. square $4k + m$ for $m \in \{0, 1, 2, 3\}$ and find the remainder of each when divided by 4 30. a. prime b. prime
 c. prime d. prime e. not prime f. not prime g. not prime h. prime 31. $b < a < c$ 32. 5, 91 33. none 34. 209
 35. a. divisible by 2, 3, 9 b. divisible by 2 c. divisible by none d. divisible by 5 e. divisible by 2, 4, 8 f. divisible by 2, 4, 11
 36. a. 192 b. 115 c. 38 d. 269 e. 154 37. 18 38. 12 39. $\{1, 7\}$ 40. 23 41. $\{1, 6\}$ 42. 4 43. 15 44. a. 0 b. 1 45. 9
 46. a. 15 b. 20 c. 6 d. 792 47. 6 48. 7 49. a. 60 b. 168 c. 403 d. 744 50. a. 60 b. 62868 c. 18000^{30} 51. 12 52. a. 6
 b. 11 c. 65 d. 6 e. 28 f. 60 53. a. 960 b. 140 c. 672 d. 144 e. 216 f. 360 54. 6 55. 4 56. a. no b. yes c. no
 d. yes e. yes 57. 2160 58. 135 59. 8 60. a. 260 b. 173 61. 71 62. 6 63. 71 64. 140 65. $(x, y) = (8, 80)$ or $(16, 40)$
 66. a. 6 b. 1 c. 2 d. 12 e. 28 67. 12 68. 240 seconds 69. 10099 70. let the two numbers be $2n$ and $2n + 2$ and show
 that their product contains a factor of 4 and 2 71. $p^2 - 1$ is the product of two consecutive even numbers (so divisible by 8)
 and one of two consecutive even numbers is also divisible by 3 72. 1 73. 4 74. factorize $x^2 - y^2$ and show that $x = y + 1$
 75. 1806 76. 61 77. factorize $n^2 - 1$ and look at the solution to question 70 78. $(a, b) = (2, 4)$ or $(6, 4)$ 79. 5

EXERCISES 1.2

1. a. 11 b. -1 c. -16 d. -1 e. -9 2. a. -192 b. 19 c. 22 d. 11 e. -92 3. a. 8 b. 24 c. 4 d. 10 e. 0 4. a. 33
 b. 14 c. -37 d. 27 e. -19 5. a. 315 b. 10 6. a. $a - b + c$ b. $-a$ 7. a. 20 b. 9 c. 8 d. 45 8. a. -8 b. -15 c. -9
 d. -15 9. 11 10. 2187 11. 103 12. 2 13. -17 14. 7 15. 22 16. $x = 5, y = 3$ 17. solve the equation for a
 or b and show that they cannot be integers 18. $(3n + 1)(n + 1)$ 19. a. $\{-8, -2, -1, 1, 2\}$ b. $\{-11, -5, -3, -1, 1, 7\}$
 c. $\{-6, -2, 0, 1\}$ 20. a. 9 b. $9 \cdot 10^{n/2-1}$ if n is even, $9 \cdot 10^{(n-1)/2}$ if n is odd c. show that the difference between the
 sum of the even-numbered digits and the sum of the odd-numbered digits is zero

EXERCISES 1.3

1. a. 22.2 b. $0.\bar{6}$ c. -0.868 d. 1.5 2. a. $\frac{51}{250}$ b. $-3\frac{11}{25}$ c. $16\frac{17}{100}$ d. $-1\frac{1367}{5000}$ 3. a. $\frac{12}{13}$ b. $-\frac{31}{5}$ c. $\frac{197}{60}$ d. $-\frac{109}{6}$
 e. $-\frac{1}{12}$ f. $\frac{1}{6}$ g. $\frac{413}{36}$ h. $\frac{55}{36}$ 4. a. 0 b. 0 c. 29 d. 2 5. a. $-\frac{96}{35}$ b. $\frac{22}{3}$ c. $\frac{6}{7}$ d. $\frac{1}{2}$ 6. $\frac{9}{2}$ 7. $\frac{3}{5}$ 8. 51



9. 4 10. 3 11. $\frac{21}{40}$ 12. $\frac{7}{16}$ 13. $-\frac{251}{19}$ 14. a. $\frac{80}{23}$ b. $-\frac{65}{27}$ c. $\frac{24}{25}$ d. $\frac{461}{17}$ 15. -10 16. $-\frac{58}{105}$ 17. a. 6
 b. 2 c. 4 d. 4 18. 6 19. $\frac{26}{35}$ 20. $m + 45$ 21. $\frac{7}{11}$ 22. $a > b > c$ 23. $\frac{26}{15}, \frac{27}{15}, \frac{28}{15}$ 24. a. $-\frac{5}{8} < -\frac{2}{5} < 0$
 b. $\frac{3}{8} < \frac{4}{9} < \frac{5}{6}$ c. $\frac{6}{11} < \frac{8}{13} < 1\frac{2}{3}$ d. $-2\frac{1}{4} < -1\frac{1}{3} < -\frac{7}{6}$ e. $-\frac{8}{3} < -\frac{7}{5} < -1\frac{1}{4} < -\frac{6}{7}$ 25. $\frac{a-1}{a} < \frac{a}{a+1} < \frac{a+1}{a+2}$
 26. $A < B$ 27. a. 1.22 b. -0.68 c. 109.04 d. 2.106 e. 9.77401 28. a. 0.6 b. -7.8 c. 3.834 d. -24.24 29. a. 2
 b. 7007 c. 17 30. 100 31. a. $-\frac{3203}{990}$ b. $\frac{1367}{30000}$ c. $\frac{780946}{9999}$ 32. 1 33. $\frac{8}{51}$ 34. 9 35. $\frac{266}{9}$ 36. $\frac{99}{100}$
 37. $\frac{9}{13}$ 38. 1 39. $\frac{2abc}{ac+bc-ab}$ 40. $\frac{xd}{x+y}$ 41. $\frac{100}{101}$ 42. 1 43. 0.99 44. $A > B$ 45. 2

EXERCISES 1.4

1. a. irrational b. irrational c. irrational d. rational e. rational f. irrational 2. a. $\{-2, -1, 2, \sqrt{9}, 17\}$
 b. $\{-2, -1, -0.5, 2, \frac{3}{5}, \sqrt{9}, 17, \frac{\sqrt{25}}{4}\}$ c. $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi\}$ 3. a.  $-3 \leq x \leq 3$ b. 
 $-2 < x < 2$ c.  $x > 2$ d.  $x \geq 3$ e.  $x < -2$
 f.  $x \leq -12$ 4. a.  $(-2, 3)$ b.  $(-0.5, 0]$
 c.  $[-7, \infty)$ d.  $(-\infty, 100)$ 5. a.  $(-2, 5)$
 b.  $(1, 5)$ c.  $(-3, 2)$ d.  $(-1, 1)$ e. $\emptyset$ 6. a. $[-3, 1], -3 \leq x \leq 1$
 b. $(-\infty, 4), x < 4$ c. $(-3, 2], -3 < x \leq 2$ d. $[-1, \infty), x \geq -1$ 7. a. they have the same sign b. they have opposite signs
 c. they have the same sign d. they have opposite signs e. they have opposite signs 8. a. they have opposite signs
 b. x is positive, y has any sign c. they have the same sign d. y is positive, x has any sign e. they are both negative
 9. a. $[0, 4)$ b. $[-1, 1)$ c. $(-8, -1]$ d. $(5, 14)$ 10. $y < z < x$ 11. $z < y < x$ 12. $b < a < c < d$
 13. a. a is negative, b has any sign b. abc^2 is positive 14. positive 15. a. subtract the right-hand side from the left-hand side to get $-10 < 0$ b. subtract the right-hand side from the left-hand side to get $(a-5)^2 + 1 > 0$

- c. subtract the right-hand side from the left-hand side to get $(a^2 - 2)^2 > 0$ 16. use $\frac{a^2b^2+1}{2} \geq ab$, etc. 17. a. 8 b. 3π
18. $-2a$ 19. $2b - 2c$ 20. $2a + b$ 21. 1 22. $8a - 1$ 23. $3x$ 24. -1 25. 1 26. $yz - xy - xz$ 27. $\frac{1}{4}$ 28. $\frac{4}{3}$ 29. $\frac{11}{6}$
30. let $\sqrt{5} = \frac{m}{n}$ where m and n are relatively prime positive integers, and then show that there are no such m and n which satisfy $m^2 = 5n^2$ 31. assume that the product is rational 32. $(3, 4)$ 33. $a < 0, a > 2$ 34. $\sqrt{\frac{xy}{z}} + \sqrt{\frac{xz}{y}} + \sqrt{\frac{yz}{x}}$
35. consider the difference between the sides of each inequality 36. a. $|x| < 20$ b. $|x - 3| \leq 1$ c. $|x + 5| \geq 4$
37. $[165.7, 171.3]$ 38. 1 39. maximum = 3, minimum = -3

EXERCISES 1.5

1. a. $q = -3, r = 3$ b. $q = 4, r = 4$ c. $q = -10, r = 1$ d. $q = 10, r = 1$ e. $q = -101, r = 9$ f. $q = -10, r = 9$ 2. a. false b. true c. false d. true e. false f. true g. true h. false 3. a. 2 b. 1 c. 1 d. 5 e. 10 f. 5 g. 8 h. does not exist i. 1 j. 0 4. a. $\{3, 4, 6, 12\}$ b. $\{3, 5, 15\}$ 5. a. 1 b. 0 c. 2 d. 1 6. a. 5 b. 5 c. 2 d. 2 e. 9 f. 6 g. 7 7. a. 2 b. 1 c. 5 d. 7 8. a. 1 b. 1 c. 2 d. 4 e. 1 f. 6 g. 1 9. a. 1 b. 2 c. 5 d. 4 e. 6 10. a. Saturday b. Wednesday c. Sunday 11. a. 4 b. 1 c. 3 d. 4 12. a. $6k + 2, k \in \mathbb{Z}$ b. $7k + 2, k \in \mathbb{Z}$ c. $2k - 1, k \in \mathbb{Z}$ 13. $23k + 8, k \in \mathbb{Z}$ 14. $7k + 2, k \in \mathbb{Z}$ 15. $7k + 2, k \in \mathbb{Z}$ 16. $6k + 1, 6k + 5, k \in \mathbb{Z}$ 17. 23 18. 5 19. 1 20. 5 21. 5 22. 58 23. 1 24. 9 25. 102 26. use $3^{6n} = (-8)^{2n}$ and $2^{6n} = 8^{2n}$ 27. let $a = mk + b$ and show that $a^2 \equiv b^2 \pmod{m}$, then generalize the result for any power of n 28. consider $a^2 + a + 1$ in modulo 3 for $a \in \{0, 1, 2\}$ (possible remainders when a number is divided by 3) 29. show that $n^9 - n^3$ is divisible by 7, 8 and 9 30. see Example 113 on page 100 31. since 41 is a prime number, $40! \equiv -1 \pmod{41}$

EXERCISES 2.1

1. a. 64 b. 81 c. -9 d. 9 e. $\frac{1}{9}$ f. 1 g. 4 h. 8 i. $\frac{1}{4}$ j. -1 k. 1 l. 1 m. $\frac{1}{4}$ n. $-\frac{1}{128}$ o. 4 p. 10000 q. 0.01 r. -100
- s. 1 t. 1 u. 1 v. -1 2. a. 1 b. $\frac{y^4}{x^4}$ c. $\frac{1}{x^3y^2}$ d. $\frac{1}{x(x-y)}$ e. $\frac{y^3}{x^3}$ f. $\frac{1}{x+y}$ g. 1 h. $\frac{1}{x^4y^5}$ i. $(x-y)^6$ j. $(\frac{x-y}{x+y})^2$
3. -6 4. a. true b. false c. true d. false e. true f. false g. true h. false 5. a. $-\frac{1}{4x^2y^3}$ b. $\frac{5x^4}{3y^7}$ c. $2y^8$ d. $\frac{x^6}{y^2}$
- e. $32x^8$ f. $\frac{5}{x^5y^3}$ g. $\frac{y^3}{3x^2}$ h. a^4b^6 i. $\frac{5m^2}{2n^{14}}$ j. a^{40} k. $\frac{y^{20}}{x^4}$ l. $(x-y)^4$ m. $\frac{y^2}{x^4}$ n. $(x-1)^7$ o. $\frac{4x^3}{3}$ p. $a^2 - b^2$ q. $\frac{z^6}{x^8}$
- r. $\frac{z^2}{x^8y^9}$ 6. a. $\frac{a}{a+1}$ b. $\frac{y-x}{xy}$ c. x d. $\frac{x+y}{y-x}$ e. $-xy$ f. $\frac{x^2+xy+y^2}{x^2y^2}$ g. $-\frac{1}{x+y}$ h. $\frac{y-x}{x^2y^2}$



7. a. $2x^{n-1}$ b. $x^{m-1}(2x^2 + 1)$ c. x^{n^2} d. x^{5n+1} e. x^{n+1} f. $x^{2m} + x^m$ g. $x^{m^2+n^2}$ 8. a. $\frac{1}{x^2}$ b. $\frac{1}{x^6}$ c. -1 9. 7 10. $-x^{n-3}$
 11. $5 \cdot 10^{-6}$ 12. 100 13. $5 \cdot 10^4$ hours 14. 128 15. $\frac{7}{25}$ 16. $\frac{162y^4}{x}$ 17. 2 18. 18 19. 1 20. 44 21. $z < y < t < x$

EXERCISES 2.2

1. a. 6 b. 22 c. $\frac{6}{5}$ d. $\frac{11}{9}$ e. $\frac{1}{4}$ f. 0.02 g. 0.07 h. 0.5 2. a. -9 b. 24 c. -0.3 d. 1.73 e. $-\frac{6}{7}$ f. $\frac{13}{9}$ 3. a. 6.32
 b. 0.77 c. 2.65 d. 2.83 e. 0.45 f. 3.67 4. $-2a$ 5. 3 6. $-x + m + y - 1$ 7. 2 8. $7 - a$ 9. $x = 0, y = 2$ 10. a. 40
 b. $3 - \sqrt{7}$ c. $\frac{3}{5}$ d. 1 e. 27 f. 1.6 11. a. $-xy$ b. $x^2y^3z^2$ c. $-yz$ d. $-xz^2$ e. $(xyz)^2$ f. xy^4z^3 12. a. $\frac{\sqrt{5}}{3}$ b. $2\sqrt{6}$
 c. $\sqrt{x^2+y^2}$ d. $x\sqrt{x^2-4}$ e. $x^2\sqrt{x+4}$ f. x^2y 13. a. $2xy\sqrt{3x}$ b. $2x^2yz\sqrt{6xz}$ c. $\frac{abzt}{4xy}\sqrt{\frac{b}{5}}$ d. $\frac{3x}{7yz}$ 14. a. $\sqrt{a^5}$
 b. $\sqrt{a^5b^3}$ c. $\sqrt{3x^9}$ d. $-\sqrt{5}$ 15. a. rational b. irrational c. irrational d. rational 16. a. true b. true 17. a. $5\sqrt{3}$ b. $\sqrt{2}$
 c. $-3\sqrt{x}$ d. $7\sqrt{x}$ e. $8\sqrt{x}$ f. $6\sqrt{3} - 2\sqrt{2}$ g. $-\sqrt{2}$ h. 0 i. $10\sqrt{3}$ j. $-2\sqrt{x}$ k. $8\sqrt{2xy}$ l. $\frac{\sqrt{6}}{6}$ m. $\frac{9\sqrt{2}}{4}$ n. $\frac{5\sqrt{2ab}}{2}$
 o. $-\frac{5\sqrt{2}}{2}$ p. $\frac{4\sqrt{3}}{3} + 3\sqrt{2}$ 18. a. $-2\sqrt{3}$ b. $-2\sqrt{5}$ c. $5\sqrt{2}$ d. $-\sqrt{30}$ e. $2z\sqrt{y}$ 19. a. $3\sqrt{3} + 3$ b. $2\sqrt{2} - 4$ c. $10 - 2\sqrt{2}$
 d. $3 + 4\sqrt{3}$ e. $6 - \sqrt{6}$ f. $x - \sqrt{x}$ g. $5\sqrt{y} - y$ h. $m + \sqrt{mn}$ i. $3\sqrt{2} - 2\sqrt{6}$ j. $\sqrt{3} + 3\sqrt{2}$ k. $4x + 6\sqrt{x} - 4$ l. $x - y$ m. 2 n. 1
 20. a. -2 b. 2 21. a. true b. false c. true d. false 22. a. $3\sqrt{2}$ b. $\sqrt{3}$ c. $\frac{\sqrt{3}-1}{2}$ d. $-\frac{4\sqrt{6}+12}{3}$ e. $\frac{x+4\sqrt{x}}{x-16}$ f. $\sqrt{3} + 2$
 g. $\frac{a-5\sqrt{a}+6}{a-4}$ h. $\sqrt{6} + \sqrt{3} - \sqrt{2} - 2$ i. $\frac{5-\sqrt{15}}{2}$ 23. a. $\sqrt{\sqrt{5}-2}$ b. $\frac{(\sqrt{a}-\sqrt{b})\sqrt{\sqrt{a}+\sqrt{b}}}{a-b}$ or $\frac{\sqrt{a-b} \cdot \sqrt{\sqrt{a}-\sqrt{b}}}{a-b}$
 c. $\frac{\sqrt{6}-7}{5}$ d. $\frac{5\sqrt{3}+7\sqrt{2}-\sqrt{6}-12}{23}$ 24. a. $\sqrt{3} + 1$ b. $\sqrt{3} + \sqrt{2}$ c. $2 - \sqrt{2}$ d. $\frac{\sqrt{14}-\sqrt{2}}{2}$ e. $2\sqrt{3}$ 25. a. 2 b. 2 c. $2\sqrt{b}$
 26. a. $2\sqrt{2}$ b. -4 c. 1 d. $\sqrt{3} + 1$ 27. $1 - \sqrt{1001}$ 28. $(\sqrt{2} + \sqrt{11}) < (\sqrt{3} + \sqrt{10}) < 5$ 29. $(\sqrt{5} + \sqrt{20} + \sqrt{60}) < (\sqrt{10} + \sqrt{30} + \sqrt{50})$
 30. a. 28 b. $\{-1, 4\}$ c. 7 d. 4 31. a. $\frac{\sqrt{ab}+b}{b}$ b. $\frac{\sqrt{bc}}{c}$ c. $-\sqrt{3}$ d. $\sqrt{1-a^2}$ 32. 6 33. $b = 9a$ 34. substitute
 $3 - \sqrt{2}$ for x in the equation 35. 39 m

EXERCISES 2.3

1. a. undefined b. -2 c. 2 d. $\frac{3}{2}$ e. $\frac{5}{2}$ f. $\frac{1}{125}$ g. $\frac{1}{9}$ h. 0.5 i. -2 2. a. $2x^2y^6$ b. $a^2y^4 \cdot \sqrt[3]{25}$ c. $2xy^5 \cdot \sqrt[4]{4}$ d. $2x^3y^4$
- e. $\sqrt[20]{xy}$ f. $\sqrt[27]{x^2y^3}$ g. x^2 h. $3a \cdot \sqrt[3]{3}$ i. $\sqrt[12]{x^3y^5}$ j. $\frac{\sqrt{3x}}{x}$ k. $\sqrt[4]{3^3(x+y)^3}$ l. $(x-a)^2$ 3. a. $x\sqrt{x}$ b. $x \cdot \sqrt[12]{x^7}$
- c. $x^5y^2 \cdot \sqrt[12]{x^9y^4}$ d. $(5x^2 - 3x + 25) \cdot \sqrt[4]{x}$ e. $\frac{1}{y} \cdot \sqrt[4]{\frac{x^2}{8}}$ 4. a. $3 \cdot \sqrt[3]{2}$ b. $\sqrt[4]{2}$ c. $5 \cdot \sqrt[4]{2} - \sqrt[3]{2}$ 5. a. $\frac{\sqrt[3]{25}}{5}$ b. $\sqrt[4]{x^3}$
- c. $\frac{\sqrt[15]{2^{13}}}{2}$ d. $4(\sqrt[3]{4} + \sqrt[3]{2} + 1)$ e. $\frac{\sqrt[3]{x^2} - \sqrt[3]{x} + 1}{x+1}$ f. $\frac{x \cdot \sqrt[6]{x^5} - \sqrt[12]{x^{11}}}{x}$ g. $\sqrt[3]{9} + \sqrt[3]{6} + \sqrt[3]{4}$ h. $\sqrt[4]{8} + \sqrt[4]{2} + \sqrt{2} + 1$ i. $\frac{\sqrt[3]{4} + \sqrt[3]{3}}{7}$
- j. $2(\sqrt[16]{3} - 1)$ 6. a. 8 b. 2 c. 2 7. repeat the procedure used in question 6 to simplify the left-hand side of each equality
8. a. 3 b. -3 c. undefined d. -2 e. -2 f. 0 g. 8 h. 16 9. a. -4 b. 4 c. 1 d. -4 e. 4 10. a. -384 b. 5 c. 22
11. a. $6x^{5/6}$ b. $5x$ c. $2x^{21/20}$ d. $x - y$ e. $0.75x^{-7/20}$ f. $8x^{-3/2}y^{6/5}$ 12. a. $\sqrt[5]{a^3}$ b. $\frac{1}{\sqrt[3]{x}}$ c. $\frac{1}{\sqrt[5]{a^4}}$ d. $\sqrt{\frac{y}{x}}$ e. $\sqrt[6]{x}$ f. $\frac{1}{\sqrt[7]{x^2}}$
- g. $\frac{1}{y \cdot \sqrt[3]{y^2}}$ h. $5a\sqrt{a}$ i. $xy\sqrt{y}$ j. $2b \cdot \sqrt[7]{8a^5b^3}$ k. $\frac{1}{x \cdot \sqrt[3]{7y}}$ l. $(a+b)\sqrt{a+b}$ 13. a. $x^{1/3}$ b. $a^{1/4}$ c. $(x^2 + y^2)^{1/2}$
- d. $-(3a)^{2/5}$ e. $(x-y)^{-1/3}$ f. x^2y^2 g. $a^{1/2} + b^{1/2}$ h. $3a^{3/4}$ i. $5xy^{2/5}$ j. $x^{6/7}y^{9/7}$ k. $x^{-2/3}y^{2/9}$ l. $a^{1/3} + (x+y^2)^{1/5}$ 14. a. $a^{1/12}$
- b. $\frac{1}{2}c^{2/3}$ 15. a. $x - (xy)^{1/2} - 6y$ b. $3a - 7(ab)^{1/2} + 2b$ c. $6x - 2x^3$ d. $6a - 3a^2$ e. $9x - 4y$ f. $9x + 6(xy)^{1/2} + y$
16. a. $a^{1/5} + b^{1/5}$ b. $a^{1/4} + b^{1/4}$ c. $2y^{1/3}$ d. $c^{1/3} - 3$ 17. a. 2 b. 1 c. 1 d. $\{2, 3\}$ 18. $-\frac{1}{4}$ 19. 2 20. $x = \frac{20}{9}, y = \frac{5}{3}$
21. 0 22. 3 23. 128 24. 1 25. $\sqrt{2}$ 26. $\sqrt[16]{64} > \sqrt[10]{7^4\sqrt{7}} > \sqrt[4]{2\sqrt{\frac{5}{4}}}$ 27. 10 28. 2 29. 9 30. 6 31. $\emptyset$ 32. $\{-\frac{3}{4}, \frac{5}{4}\}$

ANSWERS TO TESTS

TEST 1A

1. B
2. D
3. B
4. C
5. E
6. A
7. B
8. D
9. D
10. C
11. C
12. C
13. E
14. C
15. B
16. D

TEST 1B

1. B
2. A
3. E
4. C
5. C
6. A
7. B
8. C
9. B
10. B
11. B
12. D
13. A
14. E
15. B
16. C

TEST 1C

1. B
2. B
3. C
4. B
5. A
6. B
7. E
8. A
9. B
10. C
11. C
12. E
13. D
14. B
15. D
16. B

TEST 2A

1. B
2. B
3. E
4. D
5. B
6. C
7. A
8. B
9. B
10. A
11. C
12. B
13. E
14. E
15. D
16. D

TEST 2B

1. B
2. D
3. B
4. E
5. A
6. D
7. D
8. D
9. E
10. B
11. B
12. C
13. A
14. C
15. D
16. D

TEST 2C

1. A
2. D
3. D
4. D
5. C
6. B
7. C
8. E
9. C
10. E
11. C
12. B
13. B
14. B
15. A
16. A

GLOSSARY

A

absolute value: the distance between a number and zero on the number line.

B

base: a number that is raised to a power in an exponential expression. In 3^2 , 3 is the base.

bounded interval: an interval that does not have infinity as its boundary.

build up: enlarge both the numerator and the denominator of a fraction in such a way that the value of the fraction does not change.

C

complex fraction: a fraction that contains fractions on its numerator or denominator.

composite number: a positive integer that has factors other than 1 and the number itself.

congruent: identical; corresponding, in agreement

conjugate: the result of writing the sum of two terms as a difference, or vice-versa. For example, the conjugate of $a + b$ is $a - b$.

coordinate plane: a plane in which the locations of points are given by x - and y -coordinates that represent horizontal and vertical distances along the x -axis and y -axis, respectively.

D

decimal number: a number in the base 10 number system which has one or more places to the right of a decimal point.

denominator: the lower part of a fraction.

digit: each of the symbols in a number. 273 has three digits: 2, 7 and 3.

dividend: a number or quantity which is divided. In $10 \div 5$, 10 is the dividend.

divisor: a quantity that evenly divides another quantity. 3 is a divisor of 15.

E

equivalent: capable of being put into a one-to-one relationship. $3/4$ and $12/15$ are equivalent fractions.

even number: a number which is exactly divisible by 2.

exponent: a number which shows the power in an exponential expression. In 3^2 , 2 is the exponent.

exponential equation: an equation which has a variable in an exponent.

F

factor: a number that divides a given number without a remainder. 2 and 3 are factors of 6.

factorization: the process of writing an integer or polynomial as a product of its factors, or the result of this process.

fraction: an expression that shows the quotient of two quantities.

G

greatest common factor: the largest integer that divides evenly into each of a given set of numbers, often abbreviated to GCF. 6 is the greatest common factor of 30 and 18. The greatest common factor is also called the greatest common divisor.

I

improper fraction: a fraction in which the numerator is larger than or equal to the denominator.

index: a natural number which shows the degree of a root. 3 is the index in $\sqrt[3]{6}$.

integer: one of the numbers in the set $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

irrational number: any real number that cannot be expressed as a ratio of two integers. $\sqrt{2}$ and π are irrational numbers.

L

least (or lowest) common multiple: the smallest integer that is divisible by two or more given integers, often abbreviated to LCM. 12 is the least common multiple of 4 and 6.

M

mixed number: a number which consists of an integer and a fraction or decimal.

modulus: an integer that can be divided without remainder into the difference of two other integers.

multiple: The product of a real number and an integer is a multiple of the integer. 25 is a multiple of 5.

N

natural number: one of the numbers in the set $\{1, 2, 3, \dots\}$.

Also called counting number.

number line: a line on which every point represents a real number.

numerator: the upper part of a fraction.

n^{th} root: a number that when multiplied by itself n times equals a given number.

O

odd number: a number which is not exactly divisible by 2.

ordered pair: a pair of numbers of the form (a, b) in which a is the first component and b is the second component.

P

perfect square: a number which is equal to the square of a natural number. $16 = 4^2$ is a perfect square.

prime number: a positive integer which cannot be exactly divided by any positive integer other than itself and one. 2, 3, 5 and 7 are examples of prime numbers.

principal square root: the non-negative square root of a number.

proper fraction: a fraction in which the numerator is less than the denominator.

Q

quotient: the number obtained when one quantity is divided by another. In $45 \div 3 = 15$, 15 is the quotient.

R

radical: the root of a quantity, indicated by the radical sign $\sqrt{\quad}$.

radicand: the quantity under a radical sign. 3 is the radicand of $\sqrt{3}$.

rational number: the ratio of an integer to a non-zero integer.

rationalizing the denominator: the process of making an irrational denominator rational by multiplying both the numerator and denominator by the conjugate of the denominator.

real number: a number that is rational or irrational.

reduce: to simplify a fraction without changing its value.

relatively prime: Two numbers are relatively prime if there is no number greater than 1 that divides both of them evenly. 10 and 33 are relatively prime.

remainder: the number which remains when one number divides another number.

repeating decimal: a decimal in which a pattern of one or more digits repeats indefinitely, e.g. 0.353535....

S

scientific notation: a way of expressing a number in terms of a decimal number between 1 and 10 multiplied by a power of 10. 10492 is a number in normal notation (also called standard notation). $1.0492 \cdot 10^4$ is the same number in scientific notation.

square root: a number that when multiplied by itself equals a given number.

T

term: In an expression such as $3x + 4y + 13$, $3x$, $4y$ and 13 are terms.

U

unbounded interval: an interval that has infinity as its boundary.

W

whole number: one of the numbers in the set $\{0, 1, 2, 3, \dots\}$.



NUMBERS

MODULAR SYSTEM

